Answers to Exam 2 - Part 1 - Math 142 - Fall 91

(1a)

$$\left.\begin{array}{llll} f(x) & = \ln x & = \ln x \\ f^{(1)}(x) & = x^{-1} & = x^{-1} \\ f^{(2)}(x) & = -x^{-2} & = -x^{-2} \\ f^{(3)}(x) & = +2x^{-3} & = +2!\,x^{-3} \\ f^{(4)}(x) & = -6x^{-4} & = -3!\,x^{-4} \\ f^{(5)}(x) & = +24x^{-5} & = +4!\,x^{-5} \end{array}\right] \xrightarrow{a=1} \begin{array}{l} f(1) = 0 \\ f^{(1)}(1) = 1 \\ f^{(2)}(1) = -1 \\ f^{(3)}(1) = 2! \\ f^{(4)}(1) = -3! \\ f^{(5)}(z) = 4!\,z^{-5} \end{array}$$

Recall, $n=4$.

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(1)}{k!}(x-1)^k = 0 + (x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4$$

$$= \boxed{(x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}$$

$$R_4(x) = \frac{f^{(5)}(z)}{5!}(x-1)^5 = \frac{4!\,z^{-5}}{5!}(x-1)^5 = \boxed{\frac{(x-1)^5}{5z^5} \quad \text{where } z \text{ is between 1 and } x.}$$

(1b) for $.8 \le x \le 1.2$, we have [z is btw. 1 and x so $.8 \le z \le 1.2$ so $\frac{1}{1.2} \le \frac{1}{z} \le \frac{1}{.8}$]

$$|R_4(x)| = \frac{1}{5}\frac{1}{|z|^5}|x-1|^5 \le \frac{1}{5}\frac{1}{(.8)^5}(.2)^5 = \frac{1}{5}\left(\frac{.2}{.8}\right)^5 = \frac{1}{5}\left(\frac{1}{4}\right)^5 = \frac{1}{5120} \le .0002$$

$\approx .0001953$.

So 3 decimal places of accuracy

(2) Approx $\cos 47°$ to 5 dpa. Since $47° \approx 45° = \frac{\pi}{4}$, we use $\boxed{a = \frac{\pi}{4}}$ $\boxed{47° = \frac{47\pi}{180}\text{ rad.}}$

$\boxed{f(x) = \cos x}$
$f'(x) = -\sin x$
$f^2(x) = -\cos x$
$f^3(x) = \sin x$
$f^4(x) =$ repeat

$f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
$f'(\frac{\pi}{4}) = -\sqrt{2}/2$
$f^2(\frac{\pi}{4}) = -\sqrt{2}/2$
$f^3(\frac{\pi}{4}) = +\sqrt{2}/2$
$f^4(\frac{\pi}{4}) =$ repeat

note: $\frac{47\pi}{180} - \frac{\pi}{4} = \frac{2\pi}{180} = \frac{\pi}{90}$

$$R_n(x) = \frac{(\pm\cos z \text{ or } \pm\sin z)}{(n+1)!}\left(x - \frac{\pi}{4}\right)^{n+1} \quad \text{where } z \text{ is between } x \text{ and } \pi/4$$

$$\left|R_n\left(\tfrac{47\pi}{180}\right)\right| = \frac{|\pm\cos z \text{ or } \pm\sin z|}{(n+1)!}\left|\frac{47\pi}{180} - \frac{\pi}{4}\right|^{n+1} \le \frac{1}{(n+1)!}\left(\frac{2\pi}{180}\right)^{n+1} = \frac{1}{(n+1)!}\left(\frac{\pi}{90}\right)^{n+1}$$

n	$\frac{1}{(n+1)!}\left(\frac{\pi}{90}\right)^{n+1}$	
$n=2$	$\frac{1}{3!}\left(\frac{\pi}{90}\right)^3 \approx .000\,001$ so $< .000\,05$	so 4 dpa ~ not enough
$\boxed{n=3}$	$\frac{1}{4!}\left(\frac{\pi}{90}\right)^4 \approx .000\,000\,061$ so $< .000\,000\,5$	so 6 dpa ~ good enough

$$P_3(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^1 - \frac{\sqrt{2}}{2}\frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{2}\frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3$$

So

$$\cos 47° = \cos\left(\frac{47\pi}{180}\text{ rad}\right) \approx \boxed{P_3\left(\frac{47\pi}{180}\right)} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\frac{\pi}{90}\right)^1 - \frac{\sqrt{2}}{2}\frac{1}{2}\left(\frac{\pi}{90}\right)^2 + \frac{\sqrt{2}}{2}\frac{1}{6}\left(\frac{\pi}{90}\right)^3$$

$$\approx \boxed{.681998}$$

(1) Seq. (a) $\lim_{n\to\infty} (-1)^n$ DNE since it oscillates btw. 1 and -1 (b) $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$ Squeeze

(c) $\lim_{n\to\infty} (-2)^n$ DNE since it osc. from $-\infty$ to ∞. $0 \leftarrow -\left(\frac{1}{2}\right)^n \le \left(\frac{1}{2}\right)^n \le \left(\frac{1}{2}\right)^n \to 0$

(d) $\lim_{n\to\infty} n(\sin n\pi) = 0$ since for all integers n, we have $n \sin n\pi = n \cdot 0 \equiv 0$.

(e) $\lim_{n\to\infty} \frac{\cos n}{3^n} = 0$ Squeeze $0 \leftarrow -\frac{1}{3^n} \le \frac{\cos n}{3^n} \le \frac{1}{3^n} \to 0$

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(2)(a) $\lim_{x\to 0^+} \frac{\sin x}{x^2} = \text{"}\cancel{\tfrac{0}{0}}\text{"} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\cos x}{2x} = \cancel{\tfrac{1}{0}} = \boxed{\infty}$ or DNE.

(b) $\lim_{x\to 0^+} x^x = e^0 = 1$ since: Let $y = x^x$ so $\ln y = x \ln x = \frac{\ln x}{x^{-1}}$ so

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \cancel{\tfrac{\infty}{\infty}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = -\lim_{x\to 0^+} x = 0$

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3. $\int_{-1}^{1} x^{-2}\,dx = \lim_{a\to 0^-} \int_{-1}^{a} x^{-2}\,dx + \lim_{b\to 0^+} \int_{b}^{1} x^{-2}\,dx = \lim_{a\to 0^-} -x^{-1}\Big|_{-1}^{a} + \lim_{b\to 0^+} -x^{-1}\Big|_{b}^{1}$

$= \lim_{a\to 0^-} -\frac{1}{a} - -\frac{1}{-1} + \lim_{b\to 0^+} -1 - -\frac{1}{b} = +\infty - 1 - 1 + \infty = \boxed{\infty}$

NOTE: eg $\lim_{t\to 0^-} \frac{1}{t} = -\infty$ but $\lim_{t\to 0^+} \frac{1}{t} = +\infty$

4(a) $\int \frac{dx}{x(x-1)} = \int \frac{-1}{x}\,dx + \int \frac{1}{x-1}\,dx = \boxed{-\ln|x| + \ln|x-1| + C} = \boxed{\ln\left|\frac{x-1}{x}\right| + C}$

$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$ so $A + B = 0$, $-A = 1$; $B = 1$, $A = -1$

(b) $\int \frac{3x+7}{x^2+4x+5}\,dx = \int \left[\frac{\frac{3}{2}(2x+4)}{x^2+4x+5} + \frac{1}{x^2+4x+5}\right] dx = \frac{3}{2}\int \frac{dt}{t} + \int \frac{dx}{(x+1)^2+1}$

$t = x^2+4x+5$, $dt = (2x+4)\,dx$

$= \frac{3}{2}\ln|t| + \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \frac{3}{2}\ln|t| + \int d\theta = \frac{3}{2}\ln|t| + \theta + C$

$x+1 = \tan\theta$ $= 3/2 \ln|x^2+4x+5| + \tan^{-1}(x+1) + C$.

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(5) $\int \sec^5 x\,dx = \sec^3 x \tan x - 3\int \sec^3 x \tan^2 x\,dx$

$= \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1)\,dx$

$= \sec^3 x \tan x - 3\int \sec^5 x\,dx + 3\int \sec^3 x\,dx$

$u = \sec^3 x$, $dv = \sec^2 x\,dx$; $du = 3\sec^2 x(\sec x \tan x)\,dx$, $v = \tan x$

$4\int \sec^5 x\,dx = \sec^3 x \tan x + 3\int \sec^3 x\,dx \Rightarrow \int \sec^5 x\,dx = \frac{1}{4}\left[\sec^3 x \tan x + \frac{3}{2}\sec x \tan x + \frac{3}{2}\ln|\sec x + \tan x| + C\right]$

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6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow a^2y^2 = a^2b^2 - b^2x^2 = b^2(a^2 - x^2) \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$

total area $= 4 \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$ $x = a\sin\theta$, $a^2 - x^2 = a^2\cos^2\theta$

$= \frac{4b}{a}\int_{x=0}^{x=a} (a\cos\theta)(a\cos\theta)\,d\theta = 4ab\int_{x=0}^{x=a} \cos^2\theta\,d\theta = 4ab\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]\Big|_{x=0}^{x=a}$

$= 4ab\left[\frac{1}{2}\theta + \frac{1}{4}\cdot 2 \cdot \sin\theta\cos\theta\right]\Big|_{x=0}^{x=a} = 4ab\left(\frac{1}{2}\right)\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right]\Big|_{x=0}^{x=a}$

$= 2ab\left[(\sin^{-1} 1 + 0) - (\sin^{-1} 0 + 0)\right] = 2ab\left(\frac{\pi}{2} - 0\right) = \boxed{\pi ab}$

(b) as $a \to b$, total area $\to \pi a^2$ as expected since as $a \to b$ the ellipse goes to a circle of radius a.