FALL 1991 / Math 142 Answers to Exam #1

(A1) $y = \arctan x \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ (A2) $y = 2^x = e^{\ln 2^x} = e^{x \ln 2} \Rightarrow \frac{dy}{dx} = (e^{x\ln 2})(\ln 2) = (2^x)(\ln 2)$

(A3) $y = e^x \Rightarrow \frac{dy}{dx} = e^x$ (A4) $y = x^e \Rightarrow \frac{dy}{dx} = ex^{e-1}$ (power rule) (A5) $y = e^e \Rightarrow \frac{dy}{dx} = 0$ (the deriv of a constant!)

(A6) $y = x^x$ (a function raised to a function so use log. differ.) $\Rightarrow \ln y = \ln x^x = x \ln x \Rightarrow$

$\frac{1}{y}\frac{dy}{dx} = (x)(\frac{1}{x}) + (1)(\ln x) \Rightarrow \frac{dy}{dx} = y\,(1 + \ln x)$. (A7) $y = \frac{x^2 \sin x}{\sqrt{x+5}\, e^2}$ use log differ. $\Rightarrow$

$\ln y = 2\ln x + \ln(\sin x) - \frac{1}{2}\ln(x+5) - 2 \Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x} + \cot x - \frac{1}{2(x+5)}\right]$.

(A8) $ye^x + xe^y = 3 \Rightarrow \frac{dy}{dx}e^x + ye^x + e^y + xe^y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-ye^x - e^y}{e^x + xe^y}$

(B1) $\int_{-1}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{-1}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}(-1) = \left(\frac{\pi}{3}\right) - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}$

(B2) $\int \frac{\log_5 x^2}{x}\,dx = \frac{2}{\ln 5}\int \frac{\ln x}{x}\,dx = \frac{2}{\ln 5}\int u\,du = \frac{2}{\ln 5}\frac{u^2}{2} + C = \frac{(\ln x)^2}{\ln 5} + C$

$\log_5(x^2) = \frac{\ln(x^2)}{\ln 5} = \frac{2\ln x}{\ln 5}$ $u = \ln x$, $du = \frac{1}{x}dx$ $u = \sin 2x$, $du = 2\cos 2x\,dx$

(B3) $\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)(\cos 2x\,dx) = \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}\left[u - \frac{u^3}{3}\right] + C = \frac{1}{2}\left[\sin 2x - \frac{\sin^3 2x}{3}\right] + C$

(B4) $\int \frac{\cot^3 x}{\tan x}\,dx = \int \cot^4 x\,dx = \int \cot^2 x(\csc^2 x - 1)\,dx = \int \cot^2 x(\csc^2 x\,dx) - \int \cot^2 x\,dx$

$= -\int u^2\,du + \int (1 - \csc^2 x)\,dx = -\frac{1}{3}\cot^3 x + x + \cot x + C$

Let $u = \cot x$, $du = -\csc^2 x\,dx$ in 1st integral

(B5) $\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$

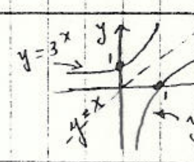
$= x\sin(\ln x) - \left[x\cos(\ln x) + \int \sin(\ln x)\,dx\right]$

$\Rightarrow \int \sin(\ln x)\,dx = \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$

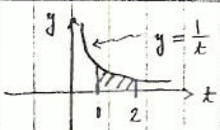
1st: $u = \sin(\ln x)$, $dv = dx$; $du = \frac{1}{x}\cos(\ln x)$, $v = x$

2nd: $u = \cos(\ln x)$, $dv = dx$; $du = -\frac{1}{x}\sin(\ln x)$, $v = x$

C. The inverse of $y = 3^x$ is the function $y = \log_3 x$



D.



shaded area $A = \int_1^2 \frac{1}{t}\,dt = \ln t \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$.

E. $P(t) = P_0 e^{kt}$. It takes 5 years for $\frac{2}{3}$ of substance to decay $\boxed{t \text{ in years}}$

i.e. after 5 years there is $\frac{1}{3}$ of the substance remaining.

$\frac{1}{3}P_0 = P(5) = P_0 e^{5k} \Rightarrow \frac{1}{3} = e^{5k} \Rightarrow \ln\frac{1}{3} = 5k \Rightarrow \boxed{\frac{1}{5}\ln\frac{1}{3} = k}$

for half-life, solve for t: $\frac{1}{2}P_0 = P(t) = P_0 e^{(\frac{1}{5}\ln\frac{1}{3})(t)}$

$\Rightarrow \frac{1}{2} = e^{\frac{t}{5}\ln\frac{1}{3}} = e^{\ln(\frac{1}{3})^{t/5}} = \left(\frac{1}{3}\right)^{t/5} \Rightarrow 2 = 3^{t/5}$

$\Rightarrow \ln 2 = \ln(3)^{t/5} = \frac{t}{5}\ln 3 \Rightarrow \boxed{t = \frac{5\ln 2}{\ln 3}}$