MARK BOX		
PROBLEM	POINTS	
1-32	96=32x3	
MML portion	4	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______

Table for Your Multiple Choice Solutions **At most ONE choice per problem.** Leave last column blank

PROBLEM							
1	1a	1b	1c	1d	1e	1f	
2	2a	2b	2c	2d	2e	2f	
3	3a	3b	3c	3d	3e	3f	
4	4a	4b	4c	4d	4e	4f	
5	5a	5b	5c	5d	5e	5f	
6	6a	6b	6c	6d	6e	6f	
7	7a	7b	7c	7d	7e	7f	
8	8a	8b	8c	8d	8e	8f	
9	9a	9b	9c	9d	9e	9f	
10	10a	10b	10c	10d	10e	10f	

continued $\Longrightarrow$

Table Continued							
11	11a	11b	11c	11d	11e	11f	
12	12a	12b	12c	12d	12e	12f	
13	13a	13b	13c	13d	13e	13f	
14	14a	14b	14c	14d	14e	14f	
15	15a	15b	15c	15d	15e	15f	
16	16a	16b	16c	16d	16e	16f	
17	17a	17b	17c	17d	17e	17f	
18	18a	18b	18c	18d	18e	18f	
19	19a	19b	19c	19d	19e	19f	
20	20a	20b	20c	20d	20e	20f	
21	21a	21b	21c	21d	21e	21f	
22	22a	22b	22c	22d	22e	22f	
23	23a	23b	23c	23d	23e	23f	
24	24a	24b	24c	24d	24e	24f	
25	25a	25b	25c	25d	25e	25f	
26	26a	26b	26c	26d	26e	26f	
27	27a	27b	27c	27d	27e	27f	
28	28a	28b	28c	28d	28e	28f	
29	29a	29b	29c	29d	29e	29f	
30	30a	30b	30c	30d	30e	30f	
31	31a	31b	31c	31d	31e	31f	
32	32a	32b	32c	32d	32e	32f	

INSTRUCTIONS

- This exam comes in two parts.
  1. STATEMENT OF MULTIPLE CHOICE PROBLEMS (pages 3–15).
     Do not hand in this part. Take this part home to learn from and to check your answers once solutions are posted.
  2. HAND IN PART (pages 1–2). Hand in only this part.
     Once you finish working the problems, raise your hand. You will be given the HAND IN PART to indicate your solutions on a table.

  Check that your copy of the exam has all 15 pages.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold during the exam (it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets.
- Cheating is grounds for a F in the course.
- At a student's request, I will project my watch upon the projector screen.
  Upon request, you will be given as much (blank) scratch paper as you need.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET):
  §8.1–8.5, 8.7, 8.8, 10.1–10.10, 11.1–11.5 .

STATEMENT OF MULTIPLE CHOICE PROBLEMS
This portion of the exam is **NOT** collected.
Thus you do not have to show your work.

- **When you initially work through the Multiple Choice Problems**,
  indicate your answers directly on the STATEMENT OF MULTIPLE CHOICE PROBLEMS portion of the exam, which is not collected.
- **After you have finished working all of the Multiple Choice Problems**,
  circle your answers in the TABLE FOR MULTIPLE CHOICE SOLUTIONS, which is collected.
- You may choice up to **1 answer** for each multiple choice problem. The scoring is as follows.
  * For a problem with precisely one answer marked and the answer is correct, 3 points.
  * All other cases, 0 points.

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  1. First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  2. Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  3. Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. Laws of Logs. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.
- Abbreviations used with Series:
  - DCT is Direct Comparison Test.
  - LCT is Limit Comparison Test.
  - AST is Alternating Series Test.

1. Compute $\displaystyle\int_1^2 \frac{x^3}{\sqrt{x^4+1}}\,dx.$

1soln. $\displaystyle\int_1^2 \frac{x^3}{\sqrt{x^4+1}}\,dx = \frac{\sqrt{17}-\sqrt{2}}{2}$

$$u = x^4+1$$
$$du = 4x^3\,dx$$
$$\tfrac{1}{4}du = x^3\,dx$$

x	u
1	2
2	17

$$\int_2^{17} \frac{1}{4}u^{-1/2}\,du = \frac{1}{4}\cdot 2u^{1/2}\Big|_2^{17}$$
$$= \frac{1}{2}\left(\sqrt{17}-\sqrt{2}\right)$$

2. Compute $\displaystyle\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx.$

2soln. $\displaystyle\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\left(\cos 1 - \cos 2\right)$

$$u = \sqrt{x}$$
$$du = \frac{1}{2\sqrt{x}}\,dx$$
$$2du = \frac{1}{\sqrt{x}}\,dx$$

x	u
1	1
4	2

$$\int_1^2 2\sin u\,du$$
$$= -2\cos u\Big|_1^2 = -2\cos 2 + 2\cos 1$$

3. Compute $\displaystyle\int_0^{\ln(2\pi)} e^x \cos\left(e^x\right)\,dx.$

3soln. Let $u = e^x$. So $du = e^x dx$. So $\int e^x \cos\left(e^x\right)\,dx = \int \cos u\,du = \sin u + C = \sin\left(e^x\right) + C$.
Next check indefinite integral: $D_x \sin\left(e^x\right) = \left[\cos\left(e^x\right)\right] D_x e^x = \left[\cos\left(e^x\right)\right] e^x$ $\boxed{\checkmark}$.
So $\int_0^{\ln(2\pi)} e^x \cos\left(e^x\right)\,dx = \sin e^x \Big|_{x=0}^{x=\ln(2\pi)} = \sin e^{\ln(2\pi)} - \sin e^0 = \sin(2\pi) - \sin 1 = 0 - \sin 1 = -\sin 1$

4. Compute $\displaystyle\int_{x=1}^{x=e} \frac{\ln x}{x^2}\,dx$.

4soln. $\displaystyle\int_{x=1}^{x=e} \frac{\ln x}{x^2}\,dx = -\frac{2}{e} + 1$. Use Integration by parts.

	u	dv	
	$\ln x$	$1/x^2$	
du	$1/x$	$-1/x$	v

$$uv - \int v\,du$$
$$= -\frac{\ln x}{x} - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx$$
$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$$
$$= -\frac{\ln x}{x} - \frac{1}{x}\Big]_1^e = \left(-\frac{1}{e} - \frac{1}{e}\right) - (0 - 1)$$
$$= -\frac{2}{e} + 1$$

3

5. Compute $\displaystyle\int_0^{\pi/4} \tan^2 x\,dx$.

5soln. $\displaystyle\int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4} \left(\sec^2 x - 1\right)\,dx = [\tan x - x]\,\Big|_0^{\pi/4} = 1 - \frac{\pi}{4}$.

6. Compute $\displaystyle\int \sec^4 x \tan^4 x\, dx$.

6soln. $\displaystyle\int \sec^4 x \tan^4 x\, dx = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

First step is to *break off your du* and so rewrite given integral as:

$$\int \sec^4 x \tan^4 x\, dx = \int \sec^2 x \tan^4 x \boxed{\sec^2 x\, dx}$$

$$\int \sec^2 x \tan^4 x \underbrace{\sec^2 x\, dx}_{du} \qquad u = \tan x \quad du = \sec^2 x\, dx$$

$$\int (1 + \tan^2 x) \tan^4 x\, \underline{\sec^2 x\, dx}$$

$$\int (1 + u^2) u^4\, du$$

$$= \int u^4 + u^6\, du$$

$$= \frac{1}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

7. What is the form for the partial fraction decomposition of $\displaystyle\frac{123x + 170000017}{x^2(x^2-4)(x^2+9)}$?

7soln. $\displaystyle\frac{123x + 170000017}{x^2(x^2-4)(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x+2} + \frac{Ex+F}{x^2+9}$

First note degree of den. $= 2 + 2 + 2 = 6 > 1 =$ degree of num. (we have *strictly bigger bottoms*) and so we do not have to dolong division.

Next factor the dem. into powers of: linear terms and irreducible quadratic terms:

$x^2(x^2-4)(x^2+9) = (x-0)^2(x-2)^1(x+2)^1(x^2+9)^1$

noting the $(x^2 + 0x + 9)$ is irreducible since $0^2 - (4)(1)(9) < 0$.

Now see the partial fraction handout from class to see that the PDF takes the form:

$\displaystyle\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x+2} + \frac{Ex+F}{x^2+9}$.

8. Compute $\displaystyle\int_0^4 \frac{x+2}{x^2+4}\, dx$.

8soln. $\displaystyle\int_0^4 \frac{x+2}{x^2+4}\, dx = \int_0^4 \left(\frac{x}{x^2+4} + \frac{2}{x^2+4}\right) dx = \left(\frac{1}{2}\ln\left|x^2+4\right| + (2)\frac{1}{2}\arctan\frac{x}{2}\right)\Big|_0^4$

$\displaystyle= \left[\frac{1}{2}\ln 20 + \arctan 2\right] - \left[\frac{1}{2}\ln 4 + \arctan 0\right] = \frac{1}{2}\ln 20 - \frac{1}{2}\ln 4 + \arctan 2.$

9. Compute $\displaystyle\int_4^6 \frac{x^2}{x-3}\,dx$.

9soln. $\displaystyle\int_4^6 \frac{x^2}{x-3}\,dx = 16 + 9\ln 3$

$$= \int_4^6 x+3+\frac{9}{x-3}\,dx$$
$$= \frac{1}{2}x^2+3x+9\ln|x-3|\Big|_4^6$$
$$= (18+18+9\ln 3)-(8+12+9\ln 1)$$
$$= 16+9\ln 3$$

$$\begin{array}{r|l} & x+3 \\ x-3 & x^2 \\ & -(x^2-3x) \\ \hline & 3x \\ & -(3x-9) \\ \hline & 9 \end{array}$$

10. What is the correct trigonometric substitution needed to evaluate the integral $\displaystyle\int \sqrt{x^2+6x+5}\,dx$?

10soln. The trig sub: $x+3 = 2\sec\theta$. Complete the square:

$$(x^2+6x+9)+5-9$$
$$(x+3)^2-4$$
$$x+3=2\sec\theta$$

11. Which of the following intergals is obtained by making the correct trig. substitution in evaluating $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$?

11soln. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int 16\sin^2\theta\,d\theta.$

$$x=4\sin\theta$$
$$dx=4\cos\theta\,d\theta$$
$$\int \frac{16\sin^2\theta\cdot 4\cos\theta}{\sqrt{16-16\sin^2\theta}}\,d\theta = \int\frac{64\sin^2\theta\cos\theta}{4\cos\theta}\,d\theta$$
$$= \int 16\sin^2\theta\,d\theta$$

12. Compute $\displaystyle\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$.

12soln. $\displaystyle\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}} = 4\sqrt{3} - \frac{4\pi}{3}$

$x = \sin\theta,\ 0 \le \theta \le \frac{\pi}{3},\ dx = \cos\theta\,d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta;$

$$\int_0^{\sqrt{3}/2} \frac{4x^2dx}{\left(1-x^2\right)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta} = 4\int_0^{\pi/3}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)d\theta = 4\int_0^{\pi/3}\left(\sec^2\theta - 1\right)d\theta = 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

13. Compute $\displaystyle\int_2^{\infty} x^3 e^{-x^4}\,dx$.

13soln. $\displaystyle\int_2^{\infty} x^3 e^{-x^4}\,dx = \frac{1}{4e^{16}}$.

$$\int_2^t x^3 e^{-x^4}\,dx \qquad u = -x^4,\ du = -4x^3\,dx$$

$$\int -\frac{1}{4}e^u\,du = -\frac{1}{4}e^u = -\frac{1}{4}e^{-x^4}\Big|_2^t$$

$$= -\frac{1}{4}e^{-t^4} + \frac{1}{4}e^{-16}$$

$$\lim_{t\to\infty}\left[-\frac{1}{4}e^{-t^4} + \frac{1}{4}e^{-16}\right] = 0 + \frac{1}{4}e^{-16}$$

14. Compute $\displaystyle\int_0^{5} \frac{dx}{x-2}$.

14soln. $\displaystyle\int_0^{5} \frac{dx}{x-2}$ diverges (i.e., The integral does not converge to a finite number.)

VA at $x = 2$. $\rightarrow \displaystyle\int_0^2 \frac{1}{x-2}\,dx + \int_2^5 \frac{1}{x-2}\,dx$

$$\int_0^t \frac{1}{x-2}\,dx = \ln|x-2|\Big|_0^t = \ln|t-2| - \ln 2$$

$$\lim_{t\to 2^-}\left[\ln|t-2| - \ln 2\right] = -\infty \qquad \underline{\text{Diverges}}$$

15. Consider the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{(-1)^n n}{3n+1}$ and $b_n = \dfrac{(-1)^n \ln n}{n}$.
Which of the following statements is true?

15soln. The sequence $\left\{\dfrac{(-1)^n n}{3n+1}\right\}_{n=1}^{\infty}$ diverges and sequence $\left\{\dfrac{(-1)^n \ln n}{n}\right\}_{n=1}^{\infty}$ converges to 0.
Since $\lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3} \neq 0$, the sequence $\left\{\frac{(-1)^n n}{3n+1}\right\}_{n=1}^{\infty}$ diverges.
Since $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$, we have that $\lim\limits_{n\to\infty} \dfrac{(-1)^n \ln n}{n} = 0$.

16. For a series $\sum\limits_{n=1}^{\infty} a_n$, the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums are $s_n = 2 + e^{1/n}$.
Which of the following statements is true?

16soln. The series $\sum a_n$ converges to 3.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2 + e^{1/n} = 2 + e^0 = 3$$

17. Which of the following series diverge by the n^{th}-term test for divergence?

(I) $\sum\limits_{n=0}^{\infty} \dfrac{2n+1}{3n+4}$

(II) $\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{4}\right)^n$

(III) $\sum\limits_{n=1}^{\infty} \dfrac{n^2+1}{n+3}$

Hint. This question is not asking which series diverge (by some test) but rather which series diverge by the n^{th}-term test.

17soln. I and III only.

(I) $\sum\limits_{n=0}^{\infty} \dfrac{2n+1}{3n+4}$ $\quad \lim\limits_{n\to\infty} \dfrac{2n+1}{3n+4} = \dfrac{2}{3} \neq 0 \rightarrow$ Diverges

(II) $\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{4}\right)^n$ $\quad \lim\limits_{n\to\infty} \left(\dfrac{1}{4}\right)^n = 0$

(III) $\sum\limits_{n=1}^{\infty} \dfrac{n^2+1}{n+3}$ $\quad \lim\limits_{n\to\infty} \dfrac{n^2+1}{n+3} = \infty \rightarrow$ Diverges

18. The series $\sum\limits_{n=1}^{\infty} \dfrac{2+\sin n}{n\sqrt{n}}$

18soln. converges by the DCT, using for comparison $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$.
Since DCT is for positive termed series, we check: $\frac{2+\sin n}{n\sqrt{n}} \geq \frac{2-1}{n\sqrt{n}} > 0$.
Also $\frac{2+\sin n}{n\sqrt{n}} \leq \frac{2+1}{n\sqrt{n}} = \frac{3}{n\sqrt{n}} = \frac{3}{n^{3/2}}$ and $\sum \frac{3}{n^{3/2}}$ (a p-series with $p = 3/2 > 1$) converges.
So bounded aboved by convergent, as needed for DCT.

19. The formal series (note: in the demoninator is the cube root $\sqrt[3]{\ }$, not the square root $\sqrt[2]{\ }$)

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{(n+1)(n+2)(n+3)}} \ .$$

is

19soln. Thinking Land: $\frac{1}{\sqrt[3]{(n+1)(n+2)(n+3)}} \overset{n \text{ big}}{\sim} \frac{1}{\sqrt[3]{(n)(n)(n)}} = \frac{1}{n}$. So let

$$b_n = \frac{1}{n} \qquad \text{and} \qquad a_n = \frac{(-1)^n}{\sqrt[3]{(n+1)(n+2)(n+3)}} \ .$$

Then

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt[3]{(n+1)(n+2)(n+3))}} = \lim_{n\to\infty} \frac{\sqrt[3]{n^3}}{\sqrt[3]{(n+1)(n+2)(n+3)}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n^3}{(n+1)(n+2)(n+3)}} = \sqrt[3]{\frac{1}{1}} = 1 \ .$$

Since $0 < \lim_{n\to\infty} \frac{|a_n|}{b_n} < \infty$, by the LCT, $\sum b_n$ and $\sum |a_n|$ *do the same thing* and we know that $\sum b_n$ is the harmonic series so $\sum b_n$ is diverges. So $\sum |a_n|$ diverges.
Now let $u_n = \frac{1}{\sqrt[3]{(n+1)(n+2)(n+3)}}$. Since $0 \le u_n \searrow 0$, by the AST, $\sum (-1)^n u_n$ converges.
Now look at the choices.

20. The series $\displaystyle\sum_{n=1}^{\infty} \frac{n\, 2^n}{3^{2n}}$ is

20soln. convergent by the Ratio Test since the limit found using the Ratio Test is $\frac{2}{9}$

$$\frac{\frac{(n+1)2^{n+1}}{3^{2(n+1)}}}{\frac{n\,2^n}{3^{2n}}} = \frac{(n+1)\,2^{n+1}}{3^{2(n+1)}} \frac{3^{2n}}{n\,2^n} = \frac{n+1}{n} \frac{2^{n+1}}{2^n} \frac{3^{2n}}{3^{2n+2}} = \frac{n+1}{n}\,(2)\,\frac{1}{3^2} \xrightarrow{n\to\infty} 1\,(2)\,\frac{1}{9} = \frac{2}{9}.$$

21. Which of the following alternating series converge by the Alternating Series Test?

(I) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(II) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(III) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$

Hint. This question is not asking which series converges (by some test) but rather which series converge by the AST.

21soln. AST says if u_n's are positive, decreasing, and $\lim_{n\to\infty} u_n = 0$ (in short, if $0 < u_n \searrow 0$), then $\sum (-1)^n u_n$ converges.
In Series (I), $u_n = \frac{1}{\sqrt{n}}$. Since $0 < \sqrt{n} \nearrow \infty$, the reciprocal satisfies $0 < \frac{1}{\sqrt{n}} \searrow 0$.
In Series (II), $u_n = \frac{1}{n}$. Since $0 < n \nearrow \infty$, the reciprocal satisfies $0 < \frac{1}{n} \searrow 0$.
In Series (III) $u_n = \frac{\ln n}{n}$. Clearly, $0 < \frac{\ln n}{n}$. Also $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} \frac{\frac{1}{n}}{1} = \frac{1}{n} = 0$. To check decreasing, let $f(x) = \frac{\ln x}{x}$ and compute: $f'(x) = \frac{\frac{1}{x}(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} < 0 \Leftrightarrow 0 > 1 - \ln x \Leftrightarrow \ln x > 1 \Leftrightarrow x > e^1$.
So for $n > e$ (so $n \ge 3$), the u_n's are decreasing.

22. Determine the radius R and the interval I of convergence for the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{5^n\, n!}$

22soln. $R = \infty$ and $I = \{-\infty, \infty\}$

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{5^{n+1}\,(n+1)!}\cdot\frac{5^n n!}{(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(x-1)^n}\cdot\frac{5^n}{5^{n+1}}\cdot\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x-1}{5\,(n+1)}\right|$$

$$= |x-1|\cdot\lim_{n\to\infty}\frac{1}{5\,(n+1)} = |x-1|\lim_{n\to\infty}\frac{1}{5}\cdot\frac{1}{n+1} = |x-1|\cdot 0 \overset{\text{Ratio Test so want}}{<} 0$$

and $|x-1|\cdot 0 < 0$ for all x.

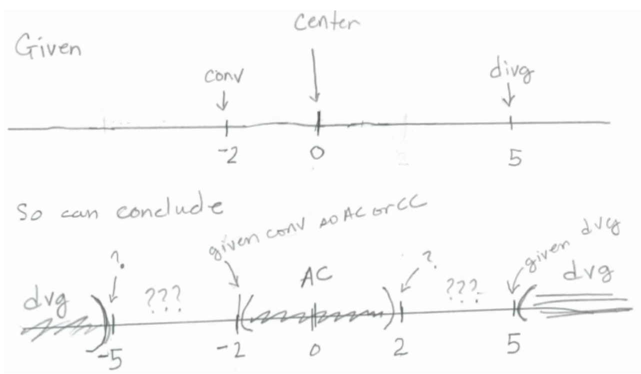
23. The power series $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -2$ and diverges when $x = 5$.

What can be concluded about the convergence and divergence below two numerical series?

(I) $\sum_{n=0}^{\infty} c_n 2^n$

(II) $\sum_{n=0}^{\infty} c_n (-6)^n$

23soln. cannot determined what (I) does but know that (II) diverges



24. Find the third order Taylor polynomial $y = p_3(x)$, centered at $x_0 = 1$, for the function $f(x) = \sqrt{x}$.

24soln. $p_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$

$f(x) = \sqrt{x}$; $f(1) = 1$

$f'(x) = \frac{1}{2}x^{-1/2}$; $f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$; $f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}x^{-5/2}$; $f'''(1) = 3/8$

$$1 + \frac{1}{2}(x-1) - \underbrace{\frac{1}{4\cdot 2!}}_{1/8}(x-1)^2 + \underbrace{\frac{3}{8\cdot 3!}}_{1/16}(x-1)^3$$

25. The Taylor series for a function $y = f(x)$, centered at $x_0 = 2$, is $\sum_{n=0}^{\infty} \frac{3n}{(n+1)!}(x-2)^n$.

What is $f^{(32)}(2)$, i.e., what is the 32^{nd} derivative of $y = f(x)$ evaluated at $x = 2$?

25soln. $f^{(32)}(2) = \frac{96}{33}$

$$\frac{f^{(n)}(2)}{n!} = \frac{3n}{(n+1)!}$$

$$\frac{f^{(32)}(2)}{32!} = \frac{3(32)}{33!}$$

$$f^{(32)}(2) = \frac{96}{33!} \cdot 32! = \frac{96}{33}$$

26. A power series representation of the function $f(x) = \frac{x}{2-x}$ is

26soln. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}}$

$$\frac{x}{2-x} = x \cdot \frac{1}{2(1-\frac{x}{2})} = \frac{x}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}}$$

27. Consider the function

$$f(x) = e^{-x} \; .$$

The 5^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_5(x) \;=\; \sum_{n=0}^{5} \frac{(-x)^n}{n!} \;=\; 1 \;-\; x \;+\; \frac{x^2}{2!} \;-\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;-\; \frac{x^5}{5!} \; .$$

The 5^{th} order Remainder term $R_5(x)$ is defined by $R_5(x) = f(x) - P_5(x)$ and so $e^{-x} \approx P_5(x)$ where the approximation is within an error of $|R_5(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_5(x)|$ that is valid for each $x \in (-1, 3)$.

27soln. $\frac{(e)(3^6)}{6!}$

For each $x \in (-1,3)$, there exists $c \in (-1,3)$ so that

$$|R_5(x)| = \left| \frac{f^{(6)}(c)}{6!} (x-0)^6 \right| = \frac{1}{6!} e^{-c} \, |x|^6 \leq \frac{1}{6!} e^{-(-1)} \, 3^6$$

28. Find an equation for the line tangent to the curve parameterized by

$x = 4 + \ln t$

$y = 2t + t^2$

at the point given by the value $t = 1$.

28soln. $y = 4x - 13$

$$\text{Slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad x(1) = \ln(1) + 4 = 4$$
$$y(1) = 2(1) + (1)^2 = 3$$

$$\frac{dy}{dt} = 2 + 2t \qquad \frac{dy}{dx} = \frac{2+2t}{1/t} \ @\ t=1 \Rightarrow \frac{2+2(1)}{1/(1)} = 4$$

$$\frac{dx}{dt} = 1/t$$

$$y - 3 = 4(x-4)$$
$$y - 3 = 4x - 16 \qquad y = 4x - 13$$

29. The curve parameterized by

$x = 1 + \cos t$

$y = -2 + \sin t$

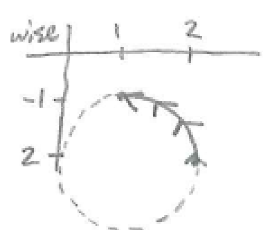
for $0 \le t \le 2\pi$ is a circle

29soln. centered at $(1, -2)$ with raduis 1, oriented counterclockwise.

$$\cos(t) = x - 1 \qquad (x-1)^2 + (y+2)^2 = 1$$
$$\sin(t) = y + 2$$

Circle, $r = 1$, $C: (1, -2)$

t	x	y
0	2	-2
$\pi/2$	1	-1



30. The arc length of the curve parameterized by

$$x = 3t^2$$
$$y = 2t^3\ ,$$

for $0 \le t \le 3$, is

30soln. ans: $20\sqrt{10} - 2$

$$\begin{aligned}
\text{AL} &= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t=0}^{t=3} \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_{t=0}^{t=3} \sqrt{6^2t^2 + 6^2t^4}\, dt \\
&= \int_{t=0}^{t=3} \sqrt{6^2t^2\,(1+t^2)}\, dt = \int_{t=0}^{t=3} 6t\sqrt{1+t^2}\, dt = 3\int_{t=0}^{t=3} (2t)\,\left(1+t^2\right)^{1/2}\, dt \\
&\underset{\text{so } du=2t\,dt}{\overset{\text{let } u=1+t^2}{=}} 3\int_{u=1}^{u=10} u^{1/2}\, du = (3)\left(\frac{2}{3}\right) u^{3/2}\,\Big|_{u=1}^{u=10} = 2u\sqrt{u}\,\Big|_{u=1}^{u=10} = 20\sqrt{10} - 2.
\end{aligned}$$

31. The point given by Cartesian coordinates $\left(-1, \sqrt{3}\right)$ has polar coordinates

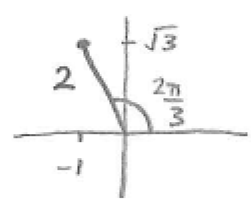
31soln. $\left(-2, \dfrac{5\pi}{3}\right)$

$r^2 = x^2 + y^2 = (-1)^2 + (\sqrt{3})^2 = 1 + 3 = 4$

$|r| = 2$

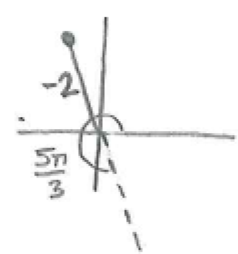
$\tan(\theta) = \dfrac{y}{x} = \dfrac{\sqrt{3}}{-1}$

$\dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{\sqrt{3}/2}{-1/2} \Rightarrow \theta = \dfrac{2\pi}{3}$



$\left(2, \frac{2\pi}{3}\right)$ Not an answer choice.

Alternative Description: $\left(-2, \frac{5\pi}{3}\right)$



32. The circle with Cartesian equation $x^2 + y^2 = 2x$ has polar equation

32soln. $r = 2\cos\theta$

$x^2 + y^2 = r^2$

$x = r\cos\theta$

$\Rightarrow r^2 = 2r\cos\theta$

Divide by r

$r = 2\cos\theta$

☺☺

Good Luck in your math fun to come!

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