MARK BOX		
PROBLEM	POINTS	
0	12	
1	10	
2	10	
3	10	
4	10	
5-16	48=12x4	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  1. (1) HAND IN PART. Hand in only this part.
  2. (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
     You can take this part home to learn from and to check your answers once the solutions are posted.
- On Problem 0, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold during the exam (it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets.
- Cheating is grounds for a F in the course.
- At a student's request, I will project my watch upon the projector screen.
  Upon request, you will be given as much (blank) scratch paper as you need.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- This exam covers (from *Calculus* by Thomas, 13th ed., ET): §10.1–10.6 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0.1. State the n^{th}**-term test** for an arbitrary series $\sum a_n$.

$\boxed{\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ (which includes the case that } \lim_{n\to\infty} a_n \text{ does not exist), then } \sum a_n \text{ diverges}}$.

0.2. Fix $r \in \mathbb{R}$ with $r \neq 1$. For $N \geq 7$, let $s_N = \sum_{\mathbf{n=7}}^{N} r^n$. (Note the sum starts at 7.) For each $N \geq 7$, the partial sums s_N can be written as: (your answer should NOT contain a "…" nor a "$\sum$" sign)

$$s_N = \boxed{\frac{r^7 - r^{N+1}}{1-r}} .$$

0.3. Geometric Sequence. $\lim_{n\to\infty} r^n = 0$ if and only if r satisfies $\boxed{|r| < 1 \text{ also ok: } -1 < r < 1 \text{ or } r \in (-1,1)}$.

0.4. Geometric Series. The series $\sum r^n$ converges if and only if r satisfies $\boxed{|r| < 1}$.

0.5. p**-series**. The series $\sum \frac{1}{n^p}$ converges if and only if $\boxed{p > 1}$.

0.6. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If $0 \leq a_n \leq c_n$ when $n \geq 17$, then $\boxed{\sum c_n \text{ converges} \implies \sum a_n \text{ converges}}$.
- If $0 \leq d_n \leq a_n$ when $n \geq 17$, then $\boxed{\sum d_n \text{ diverges} \implies \sum a_n \text{ diverges}}$.

Hint: sing the song to yourself.

0.7. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.
Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $\boxed{0 < L < \infty}$, then $[\sum b_n$ converges $\iff$ $\sum a_n$ converges $]$.

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.8. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \quad \text{or} \quad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

- If $\boxed{\rho < 1}$, then $\sum a_n$ is absolutely convergent.
- If $\boxed{\rho = 1}$, then the test is inconclusive.

0.9. The **Integral Test with Remainder Estimate** says that if $\sum a_n$ satisfies the conditions of the Integral Test for a function $y = f(x)$ with $f(n) = a_n$, then

$$0 \leq \left(\sum_{k=1}^{\infty} a_k\right) - \left(\sum_{k=1}^{N} a_k\right) \leq \boxed{\int_{x=N}^{x=\infty} f(x)\,dx} .$$

0.10. The **Alternating Series Estimation Theorem** says that if $\{u_n\}_{n=1}^{\infty}$ satisfies the conditions of the Alternating Series Test, then for $N \in \mathbb{N}$,

$$\left|\left(\sum_{k=1}^{\infty} (-1)^k u_k\right) - \left(\sum_{k=1}^{N} (-1)^k u_k\right)\right| \leq \boxed{u_{N+1}} .$$

1. Determine the behavior of the given series. Circle only one choice per series.

Series	absolutely convergent	conditionally convergent	divergent
$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$	AC	CC	[✓ DVG]
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	AC	CC	[✓ DVG]
$\sum_{n=1}^{\infty} \frac{1}{n}$	AC	CC	[✓ DVG]
$\sum_{n=1}^{\infty} \frac{1}{n^2}$	[✓ AC]	CC	DVG
$\sum_{n=1}^{\infty} \frac{1}{e^n}$	[✓ AC]	CC	DVG
$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$	AC	[✓ CC]	DVG
$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	AC	[✓ CC]	DVG
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$	AC	[✓ CC]	DVG
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	[✓ AC]	CC	DVG
$\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$	[✓ AC]	CC	DV

▶
▶
▶.

On the by-hand problems: 2,3 and 4,
justify your answer below the box and then put your answer **in** the box.
Show all your work neatly and concisely. Explicitly specify which test(s) you use.
You will be graded on the quality and correctness of your justification.

2. For a natural number $n > 2$, let

$$a_n = \frac{(n-1)!}{(3n)!}$$

2.1. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{n}{(3n+1)(3n+2)(3n+3)}}$$

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{((n+1)-1)!}{(3(n+1))!}\,\frac{(3n)!}{(n-1)!} = \frac{n!}{(n-1)!}\,\frac{(3n)!}{(3n+3)!} \\ &= \frac{(n-1)!n}{(n-1)!}\,\frac{(3n)!}{(3n)!\,(3n+1)(3n+2)(3n+3)} = \frac{n}{(3n+1)(3n+2)(3n+3)}\end{aligned}$$

2.2. Determine the behavior of the given series. You may use the previous part of this problem.

$$\sum_{n=2}^{\infty} \frac{(n-1)!}{(3n)!}$$

☒ absolutely convergent

☐ ~~conditionally convergent~~ (cannot be since it's a positive termed series)

☐ divergent

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \overset{\text{from above part}}{=} \lim_{n\to\infty}\frac{n}{(3n+1)(3n+2)(3n+3)} \\ &\overset{\textcircled{A}}{=} \lim_{n\to\infty}\end{aligned}$$

$$\left|\frac{a_{n+1}}{a_n}\right| \overset{\text{previous}}{\underset{\text{part}}{=}} \frac{n}{(3n+1)(3n+2)(3n+3)} \overset{\textcircled{A}}{=} \frac{\frac{n}{n^3}}{\frac{(3n+1)}{n}\,\frac{(3n+2)}{n}\,\frac{(3n+3)}{n}}$$

$$\overset{\textcircled{A}}{=} \frac{\frac{1}{n^2}}{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)} \xrightarrow{n\to\infty} \frac{0}{(3)(3)(3)} = 0.$$

Since $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, by the ratio test, the series $\sum a_n$ is absolutely convergent.

3. Consider the series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n+1}\right) \;-\; \cos\left(\frac{1}{n+2}\right)\right] . \qquad (3.1)$$

3.1. Find an expression for the N^{th} partial sum s_N of the series in (3.1).
Your solution should NOT contain a summation $\sum$ sign or the … sign.

$$s_N \;=\; \cos\tfrac{1}{2} - \cos\tfrac{1}{N+2}$$

$$S_n = \overset{a_1}{\left[\cos\frac{1}{2} - \underline{\cos\frac{1}{3}}\right]} + \overset{a_2}{\left[\underline{\cos\frac{1}{3}} - \cos\frac{1}{4}\right]} + \overset{a_3}{\left[\cos\frac{1}{4} - \cos\frac{1}{5}\right]} + \dots$$

$$\dots + \overset{a_{n-2}}{\left[\cos\frac{1}{n-1} - \cos\frac{1}{n}\right]} + \overset{a_{n-1}}{\left[\cos\frac{1}{n} - \cos\frac{1}{n+1}\right]} + \left[\cos\frac{1}{n+1} - \cos\frac{1}{n+2}\right]$$

Cancelling Terms,

$$\boxed{S_n = \cos\frac{1}{2} - \cos\frac{1}{n+2}}$$

3.2. If the series in (3.1) converges, then in the box write the number to which it converges.
If the series in (3.1) diverges, then in the box write "diverges".

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n+1}\right) \;-\; \cos\left(\frac{1}{n+2}\right)\right] = \cos\tfrac{1}{2} - 1$$

$$\sum_{n=1}^{\infty} \cos\frac{1}{n+1} - \cos\frac{1}{n+2} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \cos\frac{1}{2} - \cos\frac{1}{n+2} = \cos\left(\frac{1}{2}\right) - \cos(0)$$

$$= \boxed{\cos\left(\frac{1}{2}\right) - 1}$$

4. Determine the behavior of the given series.

$$\sum_{n=2}^{\infty} (-1)^n \, \frac{\ln n}{n}$$

☐ absolutely convergent

☒ conditionally convergent

☐ divergent

$|a_n| = \dfrac{\ln n}{n}$ $\qquad \dfrac{1}{n} = b_n$

$$\boxed{\lim_{n\to\infty} \frac{\ln n}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0}$$

$\lim\limits_{n\to\infty} \dfrac{|a_n|}{b_n} = \lim\limits_{n\to\infty} \dfrac{\ln n}{\cancel{n}} \cdot \dfrac{\cancel{n}}{1} = \infty$, so $\Sigma|a_n|$ diverges by LCT

AST

1) $u_n = \dfrac{\ln n}{n}$

2) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$ (see above)

3) u_n decreases

$f(n) = \dfrac{\ln n}{n}$

$f'(n) = \dfrac{n \cdot \frac{1}{n} - \ln n\,(1)}{n^2}$

$f'(n) = \dfrac{1 - \ln n}{n^2}$ any n equal to or above 2 will result in a negative value in numerator and positive in denominator, so f'(n) is always decreasing on $[2, \infty)$

So, by AST, Σa_n converges

By LCT and AST, Σa_n is conditionally convergent

MULTIPLE CHOICE PROBLEMS

The statement of the multiple choice problems, along with the scoring, can be found in the ~~Statement~~ of Multiple Choice Problems portion of the exam.

Table for Multiple Choice Solutions								Do Not Write Below			
PROBLEM							# of sol'ns circled	1	2	B	x
5	5a	5b	5c	[✓ 5d]	5e	5f					
6	[✓ 6a]	6b	6c	6d	6e	6f					
7	7a	[✓ 7b]	7c	7d	7e	7f					
8	[✓ 8a]	8b	8c	8d	8e	8f					
9	[✓ 9a]	9b	9c	9d	9e	9f					
10	10a	10b	10c	10d	[✓ 10e]	10f					
11	11a	11b	11c	[✓ 11d]	11e	11f					
12	12a	12b	[✓ 12c]	12d	12e	12f					
13	13a	[✓ 13b]	13c	13d	13e	13f					
14	14a	14b	14c	14d	[✓ 14e]	14f					
15	15a	[✓ 15b]	15c	15d	15e	15f					
16	[✓ 16a]	[✓ 16b]	[✓ 16c]	[✓ 16d]	[✓ 16e]	[✓ 16f]					
Fill in the "number of solutions circled" column with: 0, 1, or 2.											
								4	1	0	0

STATEMENT OF MULTIPLE CHOICE PROBLEMS
This portion of the exam is **NOT** collected.
Thus you do not have to show your work.

Instructions:

- **When you initially work through the Multiple Choice Problems**, indicate your answers directly on the STATEMENT OF MULTIPLE CHOICE PROBLEMS portion of the exam, which is not collected.
- **After you have finished working all of the Multiple Choice Problems**, circle your answers in the TABLE FOR MULTIPLE CHOICE SOLUTIONS, which is collected.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - $*$ For a problem with precisely one answer marked and the answer is correct, 4 points.
  - $*$ For a problem with precisely two answers marked, one of which is correct, 1 points.
  - $*$ All other cases, 0 points.

Abbreviations used:

- DCT is Direct Comparison Test.
- LCT is Limit Comparison Test.
- AST is Alternating Series Test.

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5. The $\lim\limits_{n\to\infty} \dfrac{\sqrt[2]{9n^2-8n+7}}{\sqrt[3]{8n^3+7n^2-6n-5}}$

5soln. The denominator's *dominating force* is $\sqrt[3]{n^3}=n$. So divide the numerator and denominator by n.

$$\frac{\sqrt[2]{9n^2-8n+7}}{\sqrt[3]{8n^3+7n^2-6n-5}} = \frac{\frac{\sqrt[2]{9n^2-8n+7}}{n}}{\frac{\sqrt[3]{8n^3+7n^2-6n-5}}{n}} = \frac{\sqrt[2]{\frac{9n^2-8n+7}{n^2}}}{\sqrt[3]{\frac{8n^3+7n^2-6n-5}{n^3}}} = \frac{\sqrt[2]{9-\frac{8}{n}+\frac{7}{n^2}}}{\sqrt[3]{8+\frac{7}{n}-\frac{6}{n^2}-\frac{5}{n^3}}} \xrightarrow{n\to\infty} \frac{\sqrt[2]{9-0+0}}{\sqrt[3]{8+0-0-0}} = \frac{3}{2}.$$

6. The $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$

6soln.

$\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

7. The sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \dfrac{(-1)^n\ (n+1)}{5-2n}$,

7soln.

Alternating!

$$\lim_{n\to\infty} \frac{n+1}{5-2n} = \frac{-1}{2} \neq 0$$

Since the limit is nonzero,

the alternating sequence diverges

8. The sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \dfrac{\ln n}{\ln(2n)}$,

8soln.

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(2n)} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2n}\cdot 2} \leftarrow \text{Chain Rule}$$

$$\lim_{n\to\infty} \frac{1/n}{1/n} = \lim_{n\to\infty} 1 = 1$$

Sequence converges to 1

9. Which of the following statements is always true?

I. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$. True. ($n^{\text{th}}$ term test for divergence)

II. If $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. False. (consider $\sum \frac{1}{n}$)

III. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n-1}}{7^{2n}}$ is a geometric series with ratio $r = \dfrac{4}{7}$. False. $\frac{4^{n-1}}{7^{2n}} = \frac{4^n\,4^{-1}}{(7^2)^n} = \frac{1}{4}\left(\frac{4}{7^2}\right)^n \Rightarrow r = \frac{4}{49}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 2^n}{5^{n+1}} \;=$

10soln. $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{(-2)^n}{5^n\, 5^1} = \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{-2}{5}\right)^n = \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{-2}{5}\right)^n$ is a geometric series with ratio $r = \frac{-2}{5}$.

Find an expression for $s_N - r s_N$, which results in a *cancellation heaven.*

$$s_N = \frac{1}{5}\Big[\left(\frac{-2}{5}\right)^1 + \cancel{\left(\frac{-2}{5}\right)^2} + \ldots + \cancel{\left(\frac{-2}{5}\right)^{N-1}} + \cancel{\left(\frac{-2}{5}\right)^{N}}\Big]$$

$$\swarrow \qquad\qquad \cdot\swarrow\cdot \qquad\qquad \swarrow$$

$$\left(\frac{-2}{5}\right)s_N = \frac{1}{5}\Big[\cancel{\left(\frac{-2}{5}\right)^2} + \cancel{\left(\frac{-2}{5}\right)^3} + \ldots + \cancel{\left(\frac{-2}{5}\right)^{N}} + \left(\frac{-2}{5}\right)^{N+1}\Big]$$

substract

$$\left(1-\left(\frac{-2}{5}\right)\right)s_N \overset{\textcircled{A}}{=} s_N - \left(\frac{-2}{5}\right)s_N = \frac{1}{5}\Big[\left(\frac{-2}{5}\right)^1 - \left(\frac{-2}{5}\right)^{N+1}\Big]$$

and so

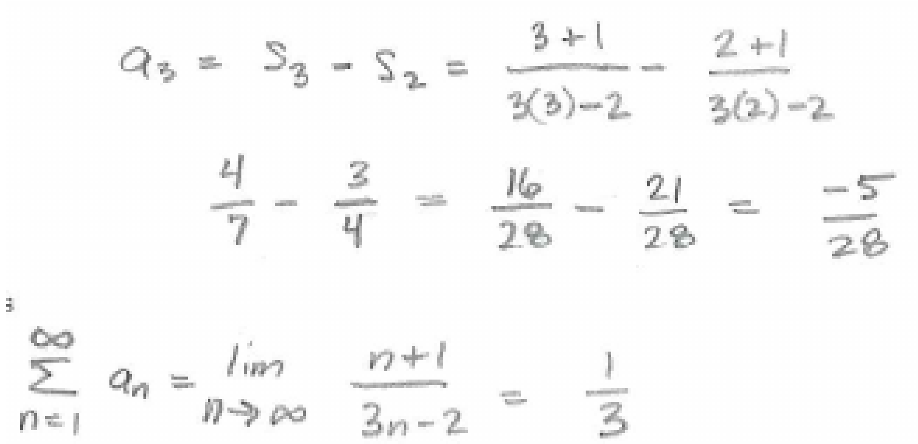
$$s_N \overset{\textcircled{A}}{=} \frac{\frac{1}{5}\left[\left(\frac{-2}{5}\right)^1 - \left(\frac{-2}{5}\right)^{N+1}\right]}{1-\left(\frac{-2}{5}\right)} \overset{\textcircled{A}}{=} \frac{\frac{1}{5}\left[\left(\frac{-2}{5}\right)^1 - \left(\frac{-2}{5}\right)^{N+1}\right]}{\frac{7}{5}} \overset{\textcircled{A}}{=} \left(\frac{5}{7}\right)\left(\frac{1}{5}\right)\left[\left(\frac{-2}{5}\right)^1 - \left(\frac{-2}{5}\right)^{N+1}\right]$$

$$\overset{\textcircled{A}}{=} \frac{1}{7}\left[\left(\frac{-2}{5}\right)^1 - \left(\frac{-2}{5}\right)^{N+1}\right] \xrightarrow{N\to\infty} \frac{1}{7}\left[\left(\frac{-2}{5}\right)^1 - 0\right] = \frac{-2}{35}$$

And so the given series converges to $\frac{-2}{35}$.

11. Suppose $s_n = \dfrac{n+1}{3n-2}$ is the sequence of partial sums for the series $\displaystyle\sum_{n=1}^{\infty} a_n$. Find a_3 and $\displaystyle\sum_{n=1}^{\infty} a_n$.

11soln.



$$a_3 = S_3 - S_2 = \frac{3+1}{3(3)-2} - \frac{2+1}{3(2)-2}$$

$$\frac{4}{7} - \frac{3}{4} = \frac{16}{28} - \frac{21}{28} = \frac{-5}{28}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{n+1}{3n-2} = \frac{1}{3}$$

12. For what values of p does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge?

12soln. Case $p > 0$. Use the AST.
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} = \sum_{n=1}^{\infty} (-1)^n u_n$ where $u_n = \frac{1}{n^p} > 0$ so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ is an alternating series.
Note $0 \le \frac{1}{n^p} \searrow 0$, i.e., $\left\{\frac{1}{n^p}\right\}_{n=1}^{\infty}$ is a postive-termed decreasing sequence which converges to zero.
So by AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges when $p > 0$.
Case $p = 0$. Use the n^{th}-term test.
$\lim_{n\to\infty} \left|\frac{(-1)^n}{n^p}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n}{n^0}\right| = \lim_{n\to\infty} 1 = 1 \neq 0$.
So $\lim_{n\to\infty} \frac{(-1)^n}{n^p}$ DNE (it's oscillating). So by the n^{th}-term test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^0}$ diverges.
Case $p < 0$. Use the n^{th}-term test. Note $-p > 0$.
$\lim_{n\to\infty} \left|\frac{(-1)^n}{n^p}\right| = \lim_{n\to\infty} n^{-p} \underset{-p>0}{\overset{\text{note}}{=}} \infty \neq 0$.
So $\lim_{n\to\infty} \frac{(-1)^n}{n^p}$ DNE (it's oscillating). So by the n^{th}-term test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ diverges when $p < 0$.

13. The series $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$

13soln. Integral Test. Define the function $f\colon [1,\infty) \to \mathbb{R}$ by $f(x) = xe^{-x^2}$. By design, $f(n) = a_n$ for each $n \in \mathbb{N}$. Clearly, $y = f(x)$ is positve and continuous on $[1,\infty)$. To check if $y = f(x)$ is decreasing, we use the 1^{st}-derivitive test. $f'(x) = 1e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1 - 2x^2) < 0 \Leftrightarrow 1 - 2x^2 < 0 \Leftrightarrow 1 < 2x^2 \Leftrightarrow \frac{1}{2} < x^2 \Leftrightarrow \frac{1}{\sqrt{2}} < |x| \overset{x\geq 0}{\Leftrightarrow} \frac{1}{\sqrt{2}} < x$. So f is decreasing when x is large enough (i.e., larger than $\frac{1}{\sqrt{2}}$). So $y = f(x)$ satisfies the conditions of the integral test.
Let $u = -x^2$: $\int xe^{-x^2}\,dx = \frac{-1}{2}\int e^{-x^2}\boxed{-2x\,dx} = \frac{-1}{2}\int e^u\,du = \frac{-1}{2}e^u + C = \frac{-1}{2}e^{-x^2} + C$.
Next calculate: $\int_{x=1}^{x=\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_{x=1}^{x=t} xe^{-x^2}\,dx = \lim_{t\to\infty} \frac{-1}{2}e^{-x^2} \Big|_{x=1}^{x=t} = \frac{-1}{2}\lim_{t\to\infty}\left[e^{-t^2} - e^{-1}\right] = \frac{-1}{2}\lim_{t\to\infty}\left[\frac{1}{e^{t^2}} - \frac{1}{e^{1^2}}\right] = \frac{-1}{2}\left[0 - \frac{1}{e}\right] = \frac{1}{2e}$. So $\int_{x=1}^{x=\infty} xe^{-x^2}\,dx$ converges to $\frac{1}{2e}$.
Since $\int_{x=1}^{x=\infty} xe^{-x^2}\,dx$ converges, by the integral test, the $\sum_{n=1}^{\infty} ne^{-n^2}$ also converges.

14. The series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n) + 3}{\sqrt{n}}$

14soln. For each $n \in \mathbb{N}$: $\frac{\cos(2n)+3}{\sqrt{n}} \geq \frac{-1+3}{\sqrt{n}} = \frac{2}{\sqrt{n}}$.
$\sum \frac{2}{\sqrt{n}} = 2\sum \frac{1}{n^{1/2}}$ (p-series, $p = \frac{1}{2} \leq 1$) diverges.
So by the DCT, $\sum_{n=1}^{\infty} \frac{\cos(2n)+3}{\sqrt{n}}$ also diverges.

15. The series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(n+1)!}$

15soln. Since

$$\lim_{n\to\infty}\left|\frac{3^{n+1}}{((n+1)+1)!}\cdot\frac{(n+1)!}{3^n}\right| = \lim_{n\to\infty}\left|\frac{3^n\cdot 3\cdot(n+1)!}{(n+2)!\cdot 3^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3\cdot(n+1)!}{(n+2)(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{3}{n+2}\right| = 0 < 1$$

the limit resulting from the Ratio Test is $\rho = 0$.
Since $\rho = 0 < 1$, the series converges absolutely by the Ratio Test.

16. What is your favorite series convergence test?

16soln. Solutions will vary so all solutions accepted. ☺