MARK BOX		
PROBLEM	POINTS	
0	20	
1-8	40=8x5	
9	10	
10	10	
11	10	
12	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
  You can take this part home to learn from and to check your answers once the solutions are posted.
- On Problem 0, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold during the exam (it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets.
- Cheating is grounds for a F in the course.
- At a student's request, I will project my watch upon the projector screen.
  Upon request, you will be given as much (blank) scratch paper as you need.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1-8.5, 8.7, 8.8 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in the blanks.

0.1. $\arcsin(-\frac{1}{2}) = \underline{\frac{-\pi}{6}}$ (Your answers should be an angle in **RADIANS**.)

0.2. Double-angle Formula. Your answer should involve trig functions of θ, and not of 2θ.

$\sin(2\theta) = \underline{2\ \sin\theta\cos\theta}$

0.3. Since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship between tangent (i.e., tan) and secant (i.e., sec) is $\underline{1 + \tan^2\theta = \sec^2\theta}$.

0.4. $\int \frac{du}{u} \overset{u \neq 0}{=} \underline{\ln\ |u|}$ + C

0.5. $\int u^n\, du \overset{n \neq -1}{=} \underline{\frac{u^{n+1}}{n+1}}$ +C

0.6. $\int e^u\, du = \underline{e^u}$ +C

0.7. $\int \cos u\, du = \underline{\sin u}$ $+ C$

0.8. $\int \sec^2 u\, du = \underline{\tan u}$ $+ C$

0.9. $\int \sec u \tan u\, du = \underline{\sec u}$ $+ C$

0.10. $\int \sin u\, du = \underline{-\cos u}$ $+ C$

0.11. $\int \csc^2 u\, du = \underline{-\cot u}$ $+ C$

0.12. $\int \csc u \cot u\, du = \underline{-\csc u}$ $+ C$

0.13. $\int \tan u\, du = \underline{\ln|\sec u| \overset{or}{=} -\ln|\cos u|}$ $+ C$

0.14. $\int \cot u\, du = \underline{-\ln|\csc u| \overset{or}{=} \ln|\sin u|}$ $+ C$

0.15. $\int \sec u\, du = \underline{\ln|\sec u + \tan u| \overset{or}{=} -\ln|\sec u - \tan u|}$ $+ C$

0.16. $\int \frac{1}{a^2+u^2}\, du \overset{a \geq 0}{=} \underline{\frac{1}{a}\ \tan^{-1}\left(\frac{u}{a}\right)}$ $+ C$

0.17. Trig sub.: (recall that the *integrand* is the function you are integrating) if the integrand involves $a^2 - u^2$, then one makes the substitution $u = \underline{a\sin\theta}$

0.18. Trig sub.: if the integrand involves $u^2 - a^2$, then one makes the substitution $u = \underline{a\sec\theta}$

0.19. Trig sub.: if the integrand involves $u^2 + a^2$, then one makes the substitution $u = \underline{a\tan\theta}$

0.20. Integration by parts formula: $\int u\, dv = \underline{uv - \int v du}$

MULTIPLE CHOICE PROBLEMS

- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ All other cases, 0 points.
- **When you initially work through the Multiple Choice Problems Part**, indicate your answers directly on the STATEMENT OF MC PROBLEMS, which is not collected.
- **When you finish (all of) the Multiple Choice Problems Part**, (come back to this page and) indicate your answers in the below table, which is collected.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	**1a**	1b	1c	1d	1e					
2	**2a**	2b	2c	2d	2e					
3	3a	**3b**	3c	3d	3e					
4	**4a**	4b	4c	4d	4e					
5	5a	5b	**5c**	5d	5e					
6	6a	**6b**	6c	6d	6e					
7	7a	7b	7c	**7d**	7e					
8	8a	**8b**	8c	8d	8e					

5	2	0	0
Extra Credit:			

9. Show all your work below the box then put answer in the box. Work in a correct logical fashion.

$$\int \sin^4(2x)\,\cos(2x)\;dx \;=\; \frac{1}{10}\sin^5(2x) \qquad + \text{C}$$

$$\int \sin^4 2x \cos 2x\, dx = \frac{1}{2}\int \sin^4 2x \cos 2x \cdot 2dx = \frac{1}{10}\sin^5 2x + C$$

$u = \sin(2x)$
$du = 2\cos(2x)\,dx$

$$\int \sin^4(2x)\cos(2x)\,dx = \frac{1}{2}\int u^4\,du = \frac{1}{2}\left[\frac{u^5}{5}\right] + c$$

$$= \frac{1}{10}(\sin^5(2x)) + c$$

10. Show all your work below the box then put answer in the box. Work in a correct logical fashion.

$$\int \frac{1}{x^2+2x}\,dx \;=\; \frac{\ln|x| - \ln|x+2|}{2} + C \qquad \text{also acceptable:} \qquad \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + \text{C}$$

Hint: $x^2 + 2x \;=\; x\,(x+2) \;=\; (x-0)\,(x+2)$.

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx;\; x = 0 \Rightarrow A = \frac{1}{2};\; x = -2 \Rightarrow B = -\frac{1}{2};$$

$$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\left[\ln|x| - \ln|x+2|\right] + C$$

Also acceptable: other (equivalent, due to log. laws) formulations.

First do PFD

$$\frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} = \frac{1/2}{x} - \frac{1/2}{x+2}$$

$$\frac{1}{x(x+2)} = \frac{A(x+2) + Bx}{x(x+2)} \rightarrow 1 = A(x+2) + Bx$$

$A + B = 0 \qquad B = -1/2$

$2A = 1 \qquad A = 1/2$

So, $\displaystyle \int \frac{1}{x^2+2x}\,dx = \int \frac{1/2}{x}\,dx - \int \frac{1/2}{x+2}\,dx$

$$= \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

11. Show all your work below the box then put answer in the box. Work in a correct logical fashion.

$$\int e^{10x} \cos 3x \, dx = \frac{e^{10x}}{109}\,(10\cos 3x + 3\sin 3x) + C \quad \text{also ok:} \quad \frac{10e^{10x}\cos 3x}{109} + \frac{3e^{10x}\sin 3x}{109} + \text{C}$$

11soln. We will use two integration by parts and the *bring to the other side* idea. For the two integration by parts, put the expontential function with either the u's both times or the dv's both times.

Indefinite Integral: Way # 1

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{10x} \qquad dv_1 = \cos 3x\, dx$$

$$du_1 = 10e^{10x}\, dx \qquad v_1 = \frac{1}{3}\sin 3x\ .$$

So by integration by parts

$$\int e^{10x} \cos 3x \, dx = \frac{1}{3}e^{10x}\sin 3x - \frac{10}{3}\int e^{10x}\sin 3x\, dx\ .$$

Now let

$$u_2 = e^{10x} \qquad dv_2 = \sin 3x\, dx$$

$$du_2 = 10e^{10x}\, dx \qquad v_2 = \frac{-1}{3}\cos 3x\ .$$

to get

$$\begin{aligned}\int e^{10x} \cos 3x \, dx &= \frac{1}{3}e^{10x}\sin 3x - \frac{10}{3}\left[\frac{-1}{3}e^{10x}\cos 3x - \frac{-10}{3}\int e^{10x}\cos 3x\, dx\right] \\ &= \frac{1}{3}e^{10x}\sin 3x + \frac{10}{3^2}e^{10x}\cos 3x - \frac{10^2}{3^2}\int e^{10x}\cos 3x\, dx\ .\end{aligned}$$

Now solving for $\int e^{10x}\cos 3x\, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{10^2}{3^2}\right]\int e^{10x} \cos 3x \, dx = \frac{1}{3}e^{10x}\sin 3x + \frac{10}{3^2}e^{10x}\cos 3x + K$$

and so

$$\begin{aligned}\int e^{10x} \cos 3x \, dx &= \left[\frac{3^2}{109}\right]\left(\frac{1}{3}e^{10x}\sin 3x + \frac{10}{3^2}e^{10x}\cos 3x + K\right) \\ &= \frac{3}{109}\,e^{10x}\sin 3x + \frac{10}{109}\,e^{10x}\cos 3x + \left[\frac{K3^2}{109}\right] \\ &= \frac{e^{10x}}{109}\,(3\sin 3x + 10\cos 3x) + \left[\frac{K3^2}{109}\right]\ .\end{aligned}$$

Thus

$$\int e^{10x} \cos 3x \, dx = \boxed{\frac{e^{10x}}{109}\,(10\cos 3x + 3\sin 3x) + C}\ .$$

Indefinite Integral: Way # 2

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$u_1 = \cos 3x \qquad dv_1 = e^{10x} dx$$
$$du_1 = -3 \sin 3x \, dx \qquad v_1 = \frac{1}{10} e^{10x} \ .$$

So, by integration by parts

$$\int e^{10x} \cos 3x \; dx = \frac{1}{10} e^{10x} \cos 3x - \frac{-3}{10} \int e^{10x} \sin 3x \, dx \ .$$

Now let

$$u_2 = \sin 3x \qquad dv_2 = e^{10x} dx$$
$$du_2 = 3 \cos 3x \, dx \qquad v_2 = \frac{1}{10} e^{10x} \ .$$

to get

$$\begin{aligned} \int e^{10x} \cos 3x \; dx &= \frac{1}{10} e^{10x} \cos 3x + \frac{3}{10} \left[\frac{1}{10} e^{10x} \sin 3x - \frac{3}{10} \int e^{10x} \cos 3x \, dx \right] \\ &= \frac{1}{10} e^{10x} \cos 3x + \frac{3}{10^2} e^{10x} \sin 3x - \frac{3^2}{10^2} \int e^{10x} \cos 3x \, dx \ . \end{aligned}$$

Now solving for $\int e^{10x} \cos 3x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{3^2}{10^2} \right] \int e^{10x} \cos 3x \; dx = \frac{1}{10} e^{10x} \cos 3x + \frac{3}{10^2} e^{10x} \sin 3x + K$$

and so

$$\begin{aligned} \int e^{10x} \cos 3x \; dx &= \left[\frac{10^2}{10^2 + 3^2} \right] \left(\frac{1}{10} e^{10x} \cos 3x + \frac{3}{10^2} e^{10x} \sin 3x + K \right) \\ &= \frac{10}{109} \, e^{10x} \cos 3x + \frac{3}{109} \, e^{10x} \sin 3x + \left[\frac{K 10^2}{10^2 + 3^2} \right] \\ &= \frac{e^{10x}}{109} \, (10 \cos 3x + 3 \sin 3x) + \left[\frac{K 10^2}{10^2 + 3^2} \right] \end{aligned}$$

Thus

$$\int e^{10x} \cos 3x \; dx = \boxed{\frac{e^{10x}}{109} \, (10 \cos 3x + 3 \sin 3x) + C} \ .$$

Indefinite Integral: Doesn't Work Way

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

12. Show all your work below the box then put answer in the box. Work in a correct logical fashion.

$$\int \sqrt{1-4x^2}\,dx \;=\; \frac{1}{4}\left[\arcsin(2x) \;+\; 2x\sqrt{1-4x^2}\,\right] + \text{C}$$

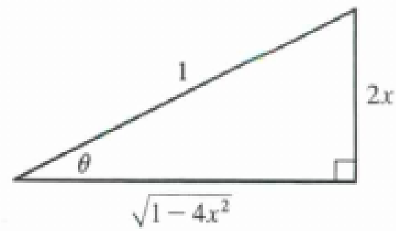
On this problem, your final answer:

- **can contain an inverve trigonmetric (i.e., arc-trig) function**
- **can not contain a trigonmetric function.**

Hint: half/double angle formula.

Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.

$$\int \sqrt{1-4x^2}\,dx = \int \cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\,\cos\theta) + C$$
$$= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\,\right] + C$$



$\frac{2x}{2} = \frac{\sin\theta}{2}$

$x = \frac{\sin\theta}{2}$

$dx = \frac{1}{2}\cos\theta\,d\theta$

$\frac{1}{2}\int\sqrt{1-4\left[\frac{\sin^2\theta}{4}\right]}\cdot\cos\theta\,d\theta$

$= \frac{1}{2}\int\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$

$= \frac{1}{2}\int\cos^2\theta\,d\theta = \frac{1}{2}\int\frac{1+\cos 2\theta}{2}\,d\theta$

$= \frac{1}{2}\left[\int\frac{1}{2}\,d\theta + \int\frac{\cos 2\theta}{2}\,d\theta\right]$ $\quad\frac{1}{2}\sin 2\theta$

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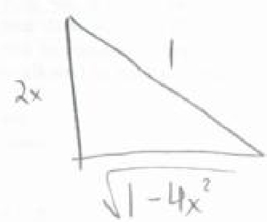
$= \frac{1}{4}\theta + \frac{1}{4}\int\cos 2\theta\,d\theta$ $\quad\theta = \sin^{-1}(2x)$



$= \frac{1}{4}\theta + \frac{1}{4}\left[\frac{1}{2}\sin 2\theta\right]$

$= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta$

$= \frac{1}{4}\theta + \frac{1}{8}[2\sin\theta\cos\theta]$

$= \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta = \frac{1}{4}\sin^{-1}(2x) + \frac{1}{4}\left[2x\cdot\frac{\sqrt{1-4x^2}}{1}\right]$

$\frac{1}{4}\sin^{-1}(2x) + \frac{x\sqrt{1-4x^2}}{2} + C$

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.
Thus you do not have to show your work.

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.

---

1. Find the polynomial $y = p\,(x)$ so that

$$\int (p\,(x))\; e^{x^2}\, dx = (x+1)\, e^{x^2} + C\ .$$

Recall that $e^{x^2} = e^{\left(x^2\right)}$. Also note that we cannot integrate the function $y = e^{x^2}$ with techniques we have learned thus far (in fact, $y = e^{x^2}$ does not have elementary antiderivative). Have you yet read the Hints at the top of page?

1soln. Since

$$D_x\left((x+1)\, e^{x^2}\right) = [D_x\,(x+1)]\, e^{x^2} + (x+1)\left[D_x e^{x^2}\right] = [1]\cdot e^{x^2} + (x+1)\cdot\left[2xe^{x^2}\right] = \left(1 + 2x^2 + 2x\right) e^{x^2}\ ,$$

by the Fundamental Theorem of Calculus,

$$\int \left(2x^2 + 2x + 1\right) e^{x^2}\, dx \;=\; (x+1)\, e^{x^2} + C\ .$$

2. Evaluate

$$\int_3^{27} \frac{1}{2x}\, dx.$$

You can use the Laws of Logs (see above Hint).

2soln.

$$\int_3^{27} \frac{1}{2x}\, dx = \frac{1}{2}\left[\int_3^{27} \frac{1}{x}\, dx\right] = \frac{1}{2}\left[\ln|x|\,\big|_3^{27}\right] = \frac{1}{2}\left[\ln 27 - \ln 3\right] = \frac{1}{2}\left[\ln \frac{27}{3}\right] = \frac{1}{2}\left[\ln 9\right] = \ln\left(9^{1/2}\right) = \ln 3.$$

►. Problems **1** and **2** were meant to reiterate the importance of the above two Hints.
Kept them in mind while doing the rest of the integration problems.

3. The integral

$$\int \frac{x}{x^2+1}\, dx$$

can be evaluated the following way.

3soln. If $u = x^2+1$, then $\int \frac{x}{x^2+1}\, dx \; = \frac{1}{2}\int \frac{2x\; dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$.

One can integrate the integral by trig. substitution using $x = \tan\theta$.

Note that the integrand is already in its Partial Fraction Decomposition.

4. Evaluate the integral

$$\int_0^4 \frac{x}{x+9}\, dx \; .$$

4soln.

$$\int \frac{x}{x+9} dx = \int 1 dx - 9\int \frac{dx}{x+9} = x - 9\ln|x+9| + C$$

$$\frac{x}{x+9} = \frac{x+9}{x+9} - \frac{9}{x+9} = 1 - \frac{9}{x+9}$$

Long Division (Fake)

Check $D_x\left[x - 9\ln|x+9|\right] = 1 - \frac{9}{x+9}$

$$= \frac{x+9}{x+9} - \frac{9}{x+9} = \frac{x}{x+9} \;. \checkmark$$

So $\int_0^4 \frac{x}{x+9}\, dx = \left[x - 9\ln|x+9|\right] \Big|_{x=0}^{x=4}$

$$= \left[4 - 9\ln|13|\right] - \left[0 - 9\ln|9|\right]$$

$$= 4 - 9\ln(13) + 9\ln(9)$$

5. Evaluate

$$\int_{x=0}^{x=1} \frac{1}{x^2+1}\, dx \; .$$

Answer:

5soln. $\int_{x=0}^{x=1} \frac{1}{x^2+1}\, dx \; = \arctan x|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

6. Evaluate

$$\int_1^2 \frac{8}{x^2-2x+2}\,dx\ .$$

Hint: complete the square in the denominator.

6soln. $x^2-2x+2=(x-1)^2+1$. So $\frac{8}{x^2-2x+2}=\frac{8}{(x-1)^2+1}$. So let $u=x-1$.

$u = x-1 \quad du = dx$

$u = 0$ when $x = 1$, $u = 1$ when $x = 2$

$$\int_1^2 \frac{8}{x^2-2x+2}\,dx = 8\int_0^1 \frac{1}{u^2+1}\,du$$

$$= 8\tan^{-1} u\Big]_0^1 = 8\left(\frac{\pi}{4}-0\right) = 2\pi$$

7. Evaluate

$$\int_{x=-1}^{x=1} \frac{1}{x^5}\,dx\ .$$

Answer:

7soln. Indefinite integral: $\int x^{-5}\,dx = \frac{x^{-4}}{-4}+C$.

Note that he function $y=x^{-5}$ is undefined at $x=0$; therefore, $\int_{-1}^1 x^{-5}dx$ is an improrper integral and we need to investigate the behaviour of $\int_{-1}^0 x^{-5}dx$ and $\int_0^1 x^{-5}dx$. Note

$$\int_0^1 x^{-5}\,dx = \lim_{b\to 0^+}\int_b^1 x^{-5}\,dx = \frac{-1}{4}\lim_{b\to 0^+}\frac{1}{x^4}\Big|_{x=b}^{x=1} = \frac{-1}{4}\lim_{b\to 0^+}\left[1-\frac{1}{b^4}\right] \overset{\cancel{(\frac{-1}{4})(-\infty)}}{=} {}^{+}\infty\ .$$

Similiarly (also can do by symmetry)

$$\int_{-1}^0 x^{-5}\,dx = \lim_{a\to 0^-}\int_{-1}^a x^{-5}\,dx = \frac{-1}{4}\lim_{a\to 0^-}\frac{1}{x^4}\Big|_{x=-1}^{x=a} = \frac{-1}{4}\lim_{a\to 0^-}\left[\frac{1}{a^4}-\frac{1}{-1}\right] \overset{\cancel{(\frac{-1}{4})(+\infty)}}{=} {}^{-}\infty\ .$$

So $\int_{-1}^1 x^{-5}\,dx$ does not exist but also does not diverge to infinity.

8. Fill in the two blanks. By comparing the improper integral

$$\int_0^\pi \frac{dt}{\sqrt{t}\ +\ \sin t}$$

to the improper integral ____________________ ,

the Direct Comparison Test (for improper integrals) gives that $\displaystyle\int_0^\pi \frac{dt}{\sqrt{t}\ +\ \sin t}$ is _________.

8soln.

$\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$. Since for $0 \le t \le \pi, 0 \le \frac{1}{\sqrt{t}+\sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^\pi \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.