MARK BOX		
PROBLEM	POINTS	
1-30	90=3x30	
Ch 11 on MML	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND-IN PART. Hand-in only this part.
  (2) NOT TO HAND-IN PART. This part will not be collected.
  Take this part home to learn from and to check your answers when the solutions are posted.
- **For the Multiple Choice** problems, circle your answer(s) on the provided chart. No need to show work.
- The MARK BOX above indicates the problems (check that you have them all) along with their points.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, give it to Prof. Girardi to hold for you during the exam (and it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- Cheating is grounds for a F in the course.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1–8.5, 8.7, 8.8, 10.1–10.10, 11.1–11.5 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

I verify that I did NOT receive help from other people or devices on the MML portion of this exam.

Signature : ______________________________

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- $\boxed{\text{You may choose up to } \mathbf{1} \text{ answers for each multiple choice problem}}$.
  The scoring is as follows.
  * For a problem with precisely one answer marked and the answer is correct, 3 points.
  * All other cases, 0 points.

At most ONE choice per problem. Table for Your Muliple Choice Solutions

PROBLEM						leave this column blank
1	1a	1b	1c	1d	1e	
2	2a	2b	2c	2d	2e	
3	3a	3b	3c	3d	3e	
4	4a	4b	4c	4d	4e	
5	5a	5b	5c	5d	5e	
6	6a	6b	6c	6d	6e	
7	7a	7b	7c	7d	7e	
8	8a	8b	8c	8d	8e	
9	9a	9b	9c	9d	9e	
10	10a	10b	10c	10d	10e	
11	11a	11b	11c	11d	11e	
12	12a	12b	12c	12d	12e	
13	13a	13b	13c	13d	13e	
14	14a	14b	14c	14d	14e	
15	15a	15b	15c	15d	15e	
16	16a	16b	16c	16d	16e	
17	17a	17b	17c	17d	17e	
18	18a	18b	18c	18d	18e	
19	19a	19b	19c	19d	19e	
20	20a	20b	20c	20d	20e	
21	21a	21b	21c	21d	21e	
22	22a	22b	22c	22d	22e	
23	23a	23b	23c	23d	23e	
24	24a	24b	24c	24d	24e	
25	25a	25b	25c	25d	25e	
26	26a	26b	26c	26d	26e	
27	27a	27b	27c	27d	27e	
28	28a	28b	28c	28d	28e	
29	29a	29b	29c	29d	29e	
30	30a	30b	30c	30d	30e	

NOT TO HAND-IN PART
STATEMENT OF MULTIPLE CHOICE PROBLEMS

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. Laws of Logs. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r\ln a$.
- Abbreviations used with Series:
  - DCT is Direct Comparison Test.
  - LCT is Limit Comparison Test.
  - AST is Alternating Series Test.

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1. Evaluate

$$\int_3^{27} \frac{1}{2x}\,dx.$$

You can use the Laws of Logs (see above Hint).

1soln.

$$\int_3^{27} \frac{1}{2x}\,dx = \frac{1}{2}\left[\int_3^{27} \frac{1}{x}\,dx\right] = \frac{1}{2}\left[\ln|x|\,\big|_3^{27}\right] = \frac{1}{2}\left[\ln 27 - \ln 3\right] = \frac{1}{2}\left[\ln\frac{27}{3}\right] = \frac{1}{2}\left[\ln 9\right] = \ln\left(9^{1/2}\right) = \ln 3.$$

2. Find the polynomial $y = p(x)$ so that

$$\int (p(x))\, e^{x^2}\,dx = xe^{x^2} + C\ .$$

Recall that $e^{x^2} = e^{\left(x^2\right)}$. Also note that we cannot integrate the function $y = e^{x^2}$ with techniques we have learned thus far (in fact, $y = e^{x^2}$ does not have elementary antiderivative). Have you yet read the Hints at the top of page?

2soln. Since

$$D_x\left(xe^{x^2}\right) = \left[D_x x\right] e^{x^2} + x\left[D_x e^{x^2}\right] = [1]\cdot e^{x^2} + x\cdot\left[2xe^{x^2}\right] = \left(1 + 2x^2\right) e^{x^2}\ ,$$

by the Fundamental Theorem of Calculus,

$$\int \left(2x^2 + 1\right) e^{x^2}\,dx \ = \ xe^{x^2} + C\ .$$

►. Problems **1** and **2** were meant to reiterate the importance of the above two Hints. Kept them in mind while doing the rest of the integration problems.

3. Evaluate
$$\int_0^1 \frac{36\,dx}{(2x+1)^3}\ .$$
Answer:

3soln. Let $u = 2x+1$. So $du = 2dx$ and we want $36dx = 18\,(2\,dx) = 18du$.
If $x = 0$ then $u = 2\,(0)+1 = 1$. If $x = 1$ then $u = 2\,(1)+1 = 3$. So

$$\int_0^1 \frac{36\,dx}{(2x+1)^3} = \int_1^3 18u^{-3}\,du = \left[\frac{18u^{-2}}{-2}\right]_1^3 = \left[\frac{-9}{u^2}\right]_1^3 = \left(\frac{-9}{3^2}\right) - \left(\frac{-9}{1^2}\right) = 8$$

4. Evaluate
$$\int_0^{\pi} \sin^2 5r\,dr\ .$$
Answer:

4soln. First let's do an indefinite integral $\int \sin^2 x\,dx$.

Solution Here we make use of half-angle identities.

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int dx - \frac{1}{4}\int (\cos 2x)(2\,dx)\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

Next let's check your answer to the indefinite integral $\int \sin^2 x\,dx$:
$$D_x\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) = \frac{1}{2} - \frac{D_x(\sin 2x)}{4} = \frac{1}{2} - \frac{(2\cos 2x)}{4} = \frac{1-\cos 2x}{2} = \sin^2 x \ \boxed{\checkmark}\ .$$
So

Let $u = 5r \Rightarrow du = 5\,dr \Rightarrow \frac{1}{5}du = dr$; $r = 0 \Rightarrow u = 0$, $r = \pi \Rightarrow u = 5\pi$

$$\int_0^{\pi} \sin^2 5r\,dr = \int_0^{5\pi} (\sin^2 u)\left(\tfrac{1}{5}du\right) = \tfrac{1}{5}\left[\tfrac{u}{2} - \tfrac{\sin 2u}{4}\right]_0^{5\pi} = \left(\tfrac{\pi}{2} - \tfrac{\sin 10\pi}{20}\right) - \left(0 - \tfrac{\sin 0}{20}\right) = \tfrac{\pi}{2}$$

5. Evaluate
$$\int_0^{\pi/2} \sin^3 x \cos^4 x\,dx\ .$$
Answer:

5soln. Since we get

$$\int \sin^3 x\cos^4 x\,dx = \int \cos^4 x\left(1-\cos^2 x\right)\sin x\,dx = \int \cos^4 x\sin x\,dx - \int \cos^6 x\sin x\,dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

$$\int_0^{\pi/2} \sin^3 x\cos^4 x\,dx = \left(\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5}\right)\Big|_0^{\pi/2} = \left(\frac{\cos^7 \frac{\pi}{2}}{7} - \frac{\cos^5 \frac{\pi}{2}}{5}\right) - \left(\frac{\cos^7 0}{7} - \frac{\cos^5 0}{5}\right)$$
$$= (0-0) - \left(\frac{1}{7} - \frac{1}{5}\right) = \frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \frac{2}{35}.$$

6. Evaluate
$$\int_0^1 x^2 e^x \, dx \ .$$
Answer:

6soln.

EXAMPLE 3 Evaluate
$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have
$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$
The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and
$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$
Using this last evaluation, we then obtain
$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx$$
$$= x^2 e^x - 2x e^x + 2e^x + C,$$
where the constant of integration is renamed after substituting for the integral on the right.

The technique of Example 3 works for any integral $\int x^n e^x \, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

$$\int_0^1 x^2 e^x \, dx = \left[x^2 e^x - 2x e^x + 2e^x\right] \big|_0^1 = [e - 2e + 2e] - [0 - 0 + 2] = e - 2 \ .$$

7. Evaluate
$$\int_1^e x^2 \ln x \, dx \ .$$
Answer:

7soln.

$$u = \ln x, du = \frac{dx}{x}; \quad dv = x^2 \, dx, v = \frac{1}{3}x^3;$$
$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \left(\frac{1}{x}\right) dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

$$\int_1^e x^2 \ln x \, dx = \left[\frac{x^3 \ln x}{3} - \frac{x^3}{9}\right] \big|_1^e = \left[\frac{e^3}{3} - \frac{e^3}{9}\right] - \left[\frac{0}{3} - \frac{1}{9}\right] = \frac{2e^3}{9} + \frac{1}{9} \ .$$

8. Evaluate
$$\int_{x=0}^{x=\pi} e^{3x} \cos 2x \ dx \ .$$
Answer:

8soln. We will use two integration by parts and the *bring to the other side* idea. For the two integration by parts, put the expontential function with either the u's both times or the dv's both times.

Indefinite Integral: Way # 1

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{3x} \qquad dv_1 = \cos 2x \, dx$$
$$du_1 = 3e^{3x} \, dx \qquad v_1 = \frac{1}{2} \sin 2x \ .$$

So by integration by parts

$$\int e^{3x} \cos 2x \ dx = \frac{1}{2} e^{3x} \sin 2x - \frac{3}{2} \int e^{3x} \sin 2x \, dx \ .$$

Now let

$$u_2 = e^{3x} \qquad dv_2 = \sin 2x \, dx$$
$$du_2 = 3e^{3x} \, dx \qquad v_2 = \frac{-1}{2} \cos 2x \ .$$

to get

$$\begin{aligned} \int e^{3x} \cos 2x \ dx &= \frac{1}{2} e^{3x} \sin 2x - \frac{3}{2} \left[\frac{-1}{2} e^{3x} \cos 2x - \frac{-3}{2} \int e^{3x} \cos 2x \, dx \right] \\ &= \frac{1}{2} e^{3x} \sin 2x + \frac{3}{2^2} e^{3x} \cos 2x - \frac{3^2}{2^2} \int e^{3x} \cos 2x \, dx \ . \end{aligned}$$

Now solving for $\int e^{3x} \cos 2x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{3^2}{2^2} \right] \int e^{3x} \cos 2x \ dx = \frac{1}{2} e^{3x} \sin 2x + \frac{3}{2^2} e^{3x} \cos 2x + K$$

and so

$$\begin{aligned} \int e^{3x} \cos 2x \ dx &= \left[\frac{2^2}{13} \right] \left(\frac{1}{2} e^{3x} \sin 2x + \frac{3}{2^2} e^{3x} \cos 2x + K \right) \\ &= \frac{2}{13} \, e^{3x} \sin 2x + \frac{3}{13} \, e^{3x} \cos 2x + \left[\frac{K2^2}{13} \right] \\ &= \frac{e^{3x}}{13} \, (2 \sin 2x + 3 \cos 2x) + \left[\frac{K2^2}{13} \right] \ . \end{aligned}$$

Thus

$$\int e^{3x} \cos 2x \ dx = \boxed{\frac{e^{3x}}{13} \, (3 \cos 2x + 2 \sin 2x) + C} \ .$$

Indefinite Integral: Way # 2

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$u_1 = \cos 2x \qquad dv_1 = e^{3x} dx$$
$$du_1 = -2 \sin 2x \, dx \qquad v_1 = \frac{1}{3} \, e^{3x} \ .$$

So, by integration by parts

$$\int e^{3x} \cos 2x \ dx = \frac{1}{3} e^{3x} \cos 2x - \frac{-2}{3} \int e^{3x} \sin 2x \, dx \ .$$

Now let

$$u_2 = \sin 2x \qquad dv_2 = e^{3x}dx$$
$$du_2 = 2\cos 2x\,dx \qquad v_2 = \frac{1}{3}\,e^{3x}\ .$$

to get

$$\begin{aligned}\int e^{3x}\ \cos 2x\ dx\ &=\ \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\left[\frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x\,dx\right]\\ &=\ \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3^2}e^{3x}\sin 2x - \frac{2^2}{3^2}\int e^{3x}\cos 2x\,dx\ .\end{aligned}$$

Now solving for $\int e^{3x}\cos 2x\,dx$ (use the *bring to the other side* idea) we get

$$\left[1+\frac{2^2}{3^2}\right]\int\ e^{3x}\ \cos 2x\ dx\ =\ \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3^2}e^{3x}\sin 2x + K$$

and so

$$\begin{aligned}\int\ e^{3x}\ \cos 2x\ dx\ &=\ \left[\frac{3^2}{3^2+2^2}\right]\left(\frac{1}{3}e^{3x}\cos 2x + \frac{2}{3^2}e^{3x}\sin 2x + K\right)\\ &=\ \frac{3}{13}\,e^{3x}\cos 2x + \frac{2}{13}\,e^{3x}\sin 2x + \left[\frac{K3^2}{3^2+2^2}\right]\\ &=\ \frac{e^{3x}}{13}\,\left(3\cos 2x + 2\sin 2x\right) + \left[\frac{K3^2}{3^2+2^2}\right]\end{aligned}$$

Thus

$$\int\ e^{3x}\ \cos 2x\ dx\ =\ \boxed{\frac{e^{3x}}{13}\,\left(3\cos 2x + 2\sin 2x\right) + C}\ .$$

Indefinite Integral: Doesn't Work Way

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

Now evaluate the definite integral.

Thus

$$\begin{aligned}\int_{x=0}^{x=\pi} e^{3x}\ \cos 2x\ dx\ &=\ \frac{e^{3x}}{13}\,\left(3\cos 2x + 2\sin 2x\right)\Big|_{x=0}^{x=\pi}\ =\ \left[\frac{e^{3\pi}}{13}\,(3+0)\right] - \left[\frac{e^{0}}{13}\,(3+0)\right]\\ &=\ \frac{3}{13}\left(e^{3\pi}-1\right)\ .\end{aligned}$$

9. Evaluate

$$\int_3^7 \frac{dx}{x^2-6x+25}\ .$$

Hint. Complete the square: $x^2 - 6x + 25 = (x\pm\ ?\ )^2 \pm\ ?$.

9soln. Complete the square:
$x^2 - 6x + 25 = (x-3)^2 + 16 = (x-3)^2 + 4^2 = u^2 + a^2$ where $u = x - 3$ and $a = 4$.
So use the substitution $u = a\tan\theta$:

$$x - 3 = 4\tan\theta \quad , \quad dx = 4\sec^2\theta\,d\theta \quad , \quad \tan\theta = \frac{x-3}{4}$$

and have

$$x^2 - 6x + 25 = (x-3)^2 + 4^2 = 4^2 \tan^2\theta + 4^2 = 4^2\left(\tan^2\theta + 1\right) = 4^2 \sec^2\theta \ .$$

So

$$\int \frac{dx}{x^2-6x+25} = \int \frac{4\sec^2\theta \, d\theta}{4^2 \sec^2\theta} = \frac{1}{4}\int d\theta = \frac{1}{4}\arctan\frac{x-3}{4} + C.$$

So

$$\int_3^7 \frac{dx}{x^2-6x+25} = \frac{1}{4}\arctan\frac{x-3}{4}\ |_3^7 = \frac{1}{4}\arctan 1 - \frac{1}{4}\arctan 0 = \frac{1}{4}\frac{\pi}{4} - 0 = \frac{\pi}{16}.$$

10. Evaluate

$$\int_3^5 \frac{\sqrt{25-x^2}}{x}\, dx \ .$$

Answer:

10soln.

Example 1 Find

$$\int \frac{\sqrt{a^2-x^2}}{x}\, dx.$$

Solution This integral is of the first type, so we write

$$x = a\sin\theta, \qquad dx = a\cos\theta\, d\theta, \qquad \sqrt{a^2-x^2} = a\cos\theta.$$

Then

$$\begin{aligned}\int \frac{\sqrt{a^2-x^2}}{x}\,dx &= \int \frac{a\cos\theta}{a\sin\theta}\, a\cos\theta\, d\theta = a\int\frac{\cos^2\theta}{\sin\theta}\, d\theta \\ &= a\int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta = a\int(\csc\theta - \sin\theta)\, d\theta \\ &= -a\ln(\csc\theta + \cot\theta) + a\cos\theta. \qquad (7)\end{aligned}$$

This completes the integration, and we now must write the answer in terms of the original variable x. We do this quickly and easily by drawing a right triangle (Fig. 10.1) whose sides are labeled in the simplest way that is consistent with the equation $x = a\sin\theta$ or $\sin\theta = x/a$. This figure tells us at once that

$$\csc\theta = \frac{a}{x}, \qquad \cot\theta = \frac{\sqrt{a^2-x^2}}{x}, \qquad \text{and} \qquad \cos\theta = \frac{\sqrt{a^2-x^2}}{a},$$

so from (7) we have

$$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) + c.$$

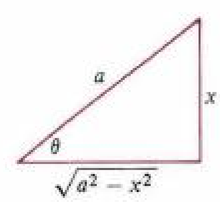


Figure 10.1

Note that, because of our limits of integration, $\csc\theta + \cot\theta \geq 0$. To see this, note that since we are integrating over $3 < x < 5$ and we are letting $\theta = \arcsin\frac{x}{5}$, we know that θ is in the 1st quadrant; thus, $\csc\theta \geq 0$ and $\cot\theta \geq 0$. So, with $a = 5$,

$$\begin{aligned}\int_3^5 \frac{\sqrt{25-x^2}}{x}\, dx &= \left[\sqrt{25-x^2} - 5\ln\left|\frac{5+\sqrt{25-x^2}}{x}\right|\,\right] |_3^5 \\ &= \left[0 - 5\ln\left|\frac{5}{5}\right|\,\right] - \left[4 - 5\ln\left|\frac{5+4}{3}\right|\,\right] = -4 + 5\ln 3 \ .\end{aligned}$$

11. Evaluate

$$\int_1^2 \frac{dx}{x\,(x+1)^2} \ .$$

Answer:

11soln.

$$\int \frac{dx}{x(x+1)^2} = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{-1}{(x+1)^2} \right) dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$$

So

$$\int_1^2 \frac{dx}{x\,(x+1)^2} = \left[\ln|x| - \ln|x+1| + \frac{1}{x+1} \right] \Big|_1^2 = \left[\ln 2 - \ln 3 + \frac{1}{3} \right] - \left[\ln 1 - \ln 2 + \frac{1}{2} \right]$$

$$= (\ln 2 - \ln 3 + \ln 2) + \left(\frac{1}{3} - \frac{1}{2} \right) = \ln \frac{4}{3} - \frac{1}{6} \ .$$

12. Evaluate

$$\int_0^1 \frac{dx}{(x+1)\ (x^2+1)} \ .$$

Answer:

12soln.

$$\frac{1}{(x+1)\left(x^2+1\right)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A\left(x^2+1\right) + (Bx+C)(x+1);\ x=-1 \Rightarrow A = \tfrac{1}{2};\ \text{coefficient of } x^2 = A+B$$

$$\Rightarrow A+B=0 \Rightarrow B = -\tfrac{1}{2};\ \text{constant } = A+C \Rightarrow A+C=1 \Rightarrow C=\tfrac{1}{2};\ \int_0^1 \frac{dx}{(x+1)\left(x^2+1\right)} = \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx$$

$$= \left[\tfrac{1}{2}\ln|x+1| - \tfrac{1}{4}\ln\left(x^2+1\right) + \tfrac{1}{2}\tan^{-1} x \right]_0^1 = \left(\tfrac{1}{2}\ln 2 - \tfrac{1}{4}\ln 2 + \tfrac{1}{2}\tan^{-1} 1 \right) - \left(\tfrac{1}{2}\ln 1 - \tfrac{1}{4}\ln 1 + \tfrac{1}{2}\tan^{-1} 0 \right)$$

$$= \tfrac{1}{4}\ln 2 + \tfrac{1}{2}\left(\tfrac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$$

13. Evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}\ .$$
Answer:

13soln.

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \tan^{-1} x \Big]_a^0 \\
&= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\
&= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\
&= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15).

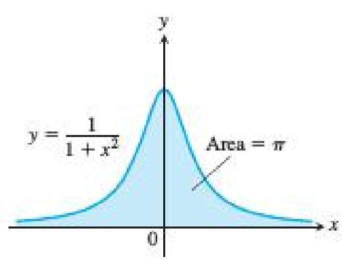


FIGURE 8.15 The area under this curve is finite (Example 2).

14. Evaluate

$$\int_{x=-1}^{x=1} \frac{1}{x^6}\, dx \ .$$

Answer:

14soln. Indefinite integral: $\int x^{-6}\, dx = \frac{x^{-5}}{-5} + C$.

Note that he function $y = x^{-6}$ is undefined at $x = 0$; therefore, $\int_{-1}^{1} x^{-6} dx$ is an improrper integral and we need to investigate the behaviour of $\int_{-1}^{0} x^{-6} dx$ and $\int_{0}^{1} x^{-6} dx$. Note

$$\int_0^1 x^{-6}\, dx = \lim_{b\to 0^+} \int_b^1 x^{-6}\, dx = \frac{-1}{5} \lim_{b\to 0^+} \frac{1}{x^5} \Big|_{x=b}^{x=1} = \frac{-1}{5} \lim_{b\to 0^+} \left[1 - \frac{1}{b^5}\right] \overset{\cancel{(\frac{-1}{5})(-\infty)}}{=} {}^{+}\infty \ .$$

Similiarly (or can just use symmetry)

$$\int_{-1}^0 x^{-6}\, dx = \lim_{a\to 0^-} \int_{-1}^a x^{-6}\, dx = \frac{-1}{5} \lim_{a\to 0^-} \frac{1}{x^5} \Big|_{x=-1}^{x=a} = \frac{-1}{5} \lim_{a\to 0^-} \left[\frac{1}{a^5} - \frac{1}{-1}\right] \overset{\cancel{(\frac{-1}{5})(-\infty)}}{=} {}^{+}\infty \ .$$

Thus

$$\int_{-1}^1 x^{-6}\, dx = \int_{-1}^0 x^{-6}\, dx + \int_0^1 x^{-6}\, dx \overset{\text{see above}}{=} \cancel{\infty + \infty} = \infty$$

and so $\int_{-1}^{1} x^{-6}\, dx$ diverges to infinity.

15. Evaluate

$$\int_{x=-1}^{x=1} \frac{1}{x^5}\, dx \ .$$

Answer:

15soln. Indefinite integral: $\int x^{-5}\, dx = \frac{x^{-4}}{-4} + C$.

Note that he function $y = x^{-5}$ is undefined at $x = 0$; therefore, $\int_{-1}^{1} x^{-5} dx$ is an improrper integral and we need to investigate the behaviour of $\int_{-1}^{0} x^{-5} dx$ and $\int_{0}^{1} x^{-5} dx$. Note

$$\int_0^1 x^{-5}\, dx = \lim_{b\to 0^+} \int_b^1 x^{-5}\, dx = \frac{-1}{4} \lim_{b\to 0^+} \frac{1}{x^4} \Big|_{x=b}^{x=1} = \frac{-1}{4} \lim_{b\to 0^+} \left[1 - \frac{1}{b^4}\right] \overset{\cancel{(\frac{-1}{4})(-\infty)}}{=} {}^{+}\infty \ .$$

Similiarly (also can do by symmetry)

$$\int_{-1}^0 x^{-5}\, dx = \lim_{a\to 0^-} \int_{-1}^a x^{-5}\, dx = \frac{-1}{4} \lim_{a\to 0^-} \frac{1}{x^4} \Big|_{x=-1}^{x=a} = \frac{-1}{4} \lim_{a\to 0^-} \left[\frac{1}{a^4} - \frac{1}{-1}\right] \overset{\cancel{(\frac{-1}{4})(+\infty)}}{=} {}^{-}\infty \ .$$

So $\int_{-1}^{1} x^{-5}\, dx$ does not exist but also does not diverge to infinity.

16. Investigate the convergence of

$$\int_{x=1}^{x=\infty} \frac{1-e^{-x}}{x}\, dx \,.$$

16soln.

EXAMPLE 9 Investigate the convergence of $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx.$

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because $\displaystyle\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$. ■

17. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{25n^3+4n^2+n-5}}{7n^{\frac{3}{2}}+6n-1} \,.$$

Answer:

17soln.

$$\frac{\sqrt{25n^3+4n^2+n-5}}{7n^{3/2}+6n-1} \overset{\uparrow}{=} \frac{\sqrt{25+\frac{4}{n}+\frac{1}{n^2}-\frac{5}{n^3}}}{7+\frac{6}{n^{1/2}}-\frac{1}{n^{3/2}}} \xrightarrow{n\to\infty} \frac{\sqrt{25}}{7} = \frac{5}{7}.$$

$\boxed{\div \text{ num \& dem. by } n^{3/2} = \sqrt{n^3}}$

18. Find **all** real numbers r satisfying that

$$\sum_{n=2}^{\infty} r^n = \frac{1}{2} \,.$$

18soln. First note that for the series $\sum_{n=2}^{\infty} r^n$ to converge (so that the problem even makes sense), we need

$$|r| < 1.$$

So let $\boxed{|r| < 1}$.

Next, to find the sum $\sum_{n=2}^{\infty} r^n$, consider the partial sums

$$s_n \overset{\text{def}}{=} r^2 + r^3 + \ldots + r^{n-1} + r^n \overset{\text{i.e.}}{=} \sum_{k=2}^{n} r^k.$$

Cancellation Heaven occurs with a geometric series when one computes $s_n - rs_n$. Let's see why.

$$\begin{aligned} s_n &= r^2 + r^3 + \ldots + r^{n-1} + r^n \\ r\,s_n &= r^3 + r^4 + \ldots + r^n + r^{n+1} \end{aligned}$$

Do you see the cancellation that would occur if we take $s_n - rs_n$?

$$\begin{array}{rcl} s_n & = & r^2 + \cancel{r^3} + \ldots + \cancel{r^{n-1}} + \cancel{r^n} \\ & & \swarrow \quad \cdot\swarrow\cdot \quad \swarrow \\ r\,s_n & = & \cancel{r^3} + \cancel{r^4} + \ldots + \cancel{r^n} + r^{n+1} \end{array}$$

substract

$$(1-r)\,s_n \overset{A}{=} s_n - rs_n = r^2 - r^{n+1}$$

So, since $r \neq 1$ (recall we have already noted that we must have $|r| < 1$),

$$s_n = \frac{r^2 - r^{n+1}}{1-r} .$$

Since $|r| < 1$, we have that $\lim_{n\to\infty} r^n = 0$. So

$$\sum_{k=2}^{n} r^k \overset{\text{def}}{=} s_n = \frac{r^2 - r^{n+1}}{1-r} \xrightarrow{n\to\infty} \frac{r^2}{1-r} = \sum_{n=2}^{\infty} r^n .$$

in other words,

$$\sum_{n=2}^{\infty} r^n \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=2}^{n} r^k \overset{\text{i.e.}}{=} \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{r^2 - r^{n+1}}{1-r} = \frac{r^2}{1-r} .$$

So we are looking for $r \in \mathbb{R}$ so that $|r| < 1$ and

$$\frac{r^2}{1-r} = \frac{1}{2}.$$

Note $\left[\frac{r^2}{1-r} = \frac{1}{2}\right] \Leftrightarrow [2r^2 = 1 - r] \Leftrightarrow [2r^2 + r - 1 = 0] \Leftrightarrow$

$$r = \frac{-1 \pm \sqrt{1 + 4\,(2)}}{2\,(2)} = \frac{-1 \pm 3}{2\,(2)} = \begin{cases} \frac{-1+3}{2(2)} = \frac{1}{2} \\ \frac{-1-3}{2(2)} = -1. \end{cases}$$

19. Consider the following two series.

Series A is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Series B is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

19soln.

- $\sum \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ so $\sum \frac{1}{n}$ diverges.
- Now consider the alternating series $\sum \frac{(-1)^n}{n}$ and set $u_n = \frac{1}{n}$ Note
  (1) $u_n = \frac{1}{n} > 0$ for each $n \in \mathbb{N}$
  (2) $u_n = \frac{1}{n} > \frac{1}{n+1} = u_{n+1}$ for each $n \in \mathbb{N}$ (i.e. $\{u_n\}_n$ is decreasing)

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

So by AST, $\sum \frac{(-1)^n}{n}$ converges.

- So $\sum \frac{1}{n}$ diverges while $\sum \frac{(-1)^n}{n}$ converges conditionally.

20. The formal series (note: in the demoninator is the cube root $\sqrt[3]{\ }$, not the square root $\sqrt[2]{\ }$)

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{(n+1)(n+2)(n+3)}} \ .$$

is

20soln. Thinking Land:

$$\frac{1}{\sqrt[3]{(n+1)(n+2)(n+3)}} \overset{n \text{ big}}{\sim} \frac{1}{\sqrt[3]{(n)(n)(n)}} = \frac{1}{n}.$$

So let

$$b_n = \frac{1}{n} \qquad \text{and} \qquad a_n = \frac{(-1)^n}{\sqrt[3]{(n+1)(n+2)(n+3)}} \ .$$

Then

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt[3]{(n+1)(n+2)(n+3))}} = \lim_{n\to\infty} \frac{\sqrt[3]{n^3}}{\sqrt[3]{(n+1)(n+2)(n+3)}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n^3}{(n+1)(n+2)(n+3)}} = \sqrt[3]{\frac{1}{1}} = 1 \ .$$

Since $0 < \lim_{n\to\infty} \frac{|a_n|}{b_n} < \infty$, by the LCT, $\sum b_n$ and $\sum |a_n|$ *do the same thing* and we know that $\sum b_n$ is the harmonic series so $\sum b_n$ is diverges. So $\sum |a_n|$ diverges.
Now let $u_n = \frac{1}{\sqrt[3]{(n+1)(n+2)(n+3)}}$. Since $0 \le u_n \searrow 0$, by the AST, $\sum (-1)^n u_n$ converges.
Now look at the choices.

21. The formal series

$$\sum_{n=17}^{\infty} \frac{1}{n \, \ln n}$$

is:

21soln. Consider $f(x) = \frac{1}{x \ln x}$ for $x \in [17, \infty)$ so that $f(n) = \frac{1}{n \ln n}$ for each $n \in \mathbb{N}$ with $n \ge 17$. Clearly, $y = f(x)$ is positive, continuous, and decreasing on $[17, \infty)$ So the conditions of the integral test are satisfied. Note that the computation of

$$\int \frac{1}{x \, \ln x} \, dx \ = \ \ln |\ln |x|| \ + \ C \ ,$$

as seen by using a u-du subsitution with $u = \ln x$. Thus

$$\int_{17}^{\infty} \frac{1}{x \, \ln x} \, dx \ = \ \lim_{b\to\infty} \int_{17}^{b} \frac{1}{x \, \ln x} \, dx \ = \ \lim_{b\to\infty} \ln |\ln |x|| \, |_{17}^{b} = \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 17)] = \infty \ .$$

So by the Integral Test, $\displaystyle\sum_{n=17}^{\infty} \frac{1}{n \, \ln n} = \infty$

Let's see what happens if we try the ratio test. Let $a_n = \frac{1}{n \ln n}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \ = \ \lim_{n\to\infty} \frac{1}{(n+1) \, \ln(n+1)} \, \frac{n \, \ln n}{1} \ = \ \lim_{n\to\infty} \frac{n}{n+1} \, \frac{\ln n}{\ln(n+1)} \ . \tag{21.1}$$

Note

$$\lim_{n\to\infty} \frac{n}{n+1} \overset{Ⓐ}{=} \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n+1}{n}} \overset{Ⓐ}{=} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \tag{21.2}$$

and (by L'Hopital's Rule)

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1,. \tag{21.3}$$

Combining (21.1) and (21.2) and (21.3) we see

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \; .$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the Ratio Test is inconclusive.

22. Consider the formal series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, n!}{(3n)!} \; .$$

Let

$$a_n = \frac{(-1)^n \, n!}{(3n)!} \qquad \text{and} \qquad \rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \; .$$

Then

22soln.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{[3\,(n+1)]!} \, \frac{(3n)!}{n!} = \frac{(n+1)!}{n!} \, \frac{(3n)!}{(3n+3)!} = \frac{n! \; (n+1)}{n!} \, \frac{(3n)!}{(3n)! \; (3n+1)\,(3n+2)\,(3n+3)}$$

$$= \frac{n+1}{(3n+1)\,(3n+2)\,(3n+3)} \xrightarrow{n\to\infty} 0 = \rho.$$

23. The formal series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.23}}$$

is

23soln.

* useful intuition!

$\ln n \leq n^q \leq e^n$; for $q > 0$ and some value of n large enough.

$$\frac{\ln n}{n^{1.23}} \leq \frac{n^q}{n^{1.23}} \quad ; \quad \frac{n^q}{n^{1.23}} = \frac{1}{n^{1.23-q}}$$

$\boxed{\text{USE DIRECT COMPARISON TEST}}$

- $\sum a_n$ is bounded above by $\sum \frac{1}{n^{1.23-q}}$;
- to converge, $\sum \frac{1}{n^{1.23-q}}$, a p-series, $1.23 - q = p > 1$

$1.23 - q > 1 \Rightarrow -q > -0.23 \Rightarrow q < 0.23$

- $q = 0.22$

Since $\frac{\ln n}{n^{1.23}} \leq \frac{1}{n^{1.01}}$

and $\sum \frac{1}{n^{1.01}}$ is a p-series w/ $p > 1$

it follows that $\sum \frac{\ln n}{n^{1.23}}$ also converges.

nice!

24. Let c be a natural number (i.e., $c \in \{1, 2, 3, 4, \ldots\}$). The series

$$\sum_{n=1}^{\infty} \frac{(n!)^6}{(cn)!}$$

24soln. Let c be a natural number (i.e., $c \in \{1, 2, 3, 4, \ldots\}$).
The series $\sum_{n=1}^{\infty} \frac{(n!)^6}{(cn)!} \frac{(n!)^6}{(cn)!}$ diverges when $c < 6$ and converges when $c \geq 6$.

Let $a_n = \frac{(n!)^6}{(cn)!}$ and $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{[(n+1)!]^6}{[n!]^6}\,\frac{(cn)!}{(cn+c)!} = \left[\frac{n!\,(n+1)}{n!}\right]^6\left[\frac{(cn)!}{(cn)!(cn+1)(cn+2)\cdots(cn+c)}\right]$$
$$= \frac{(n+1)^6}{(cn+1)(cn+2)\cdots(cn+c)} = \frac{n^6 + (\text{a poly. of degree atmost 5})}{c^c\,(n^c) + (\text{a poly. of degree at most (c-1)})}\,.$$

If $c < 6$, then $\rho = \infty$. If $c > 6$, then $\rho = 0$, If $c = 6$, then $\rho = \frac{1}{6^6} < 1$. Now apply the ratio test.

25. What is the LARGEST interval for which the formal power series

$$\sum_{n=1}^{\infty} \frac{(5x+15)^n}{4^n}$$

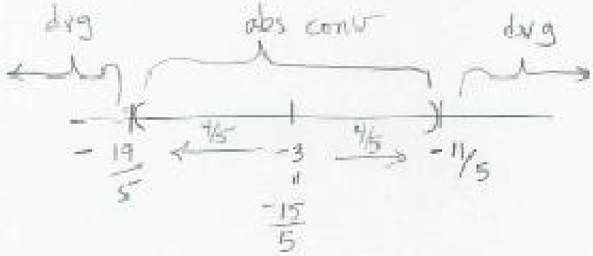
is absolutely convergent?

25soln.

$$\sum \frac{(5x+15)^n}{4^n} = \sum \frac{[5(x+3)]^n}{4^n} = \sum \left(\frac{5}{4}\right)^n (x - -3)^n \quad . \Rightarrow \text{center} =$$

$$\left[\left|\frac{(5x+15)^n}{4^n}\right|\right]^{1/n} = \left|\frac{5x+15}{4}\right| = \frac{5}{4}|x+3| \xrightarrow{n\to\infty} \frac{5}{4}|x+3|$$

$\frac{5}{4}|x+3| < 1 \iff |x+3| < \frac{4}{5} \Rightarrow$ rad. of conv. is $\frac{4}{5}$



Check endpts

$x = -11/5$: $\sum \frac{(5x+15)^n}{4^n} = \sum \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1$ $\underline{\text{dvg}}$ (to ∞)

$x = -19/5$: $\sum \frac{(5x+15)^n}{4^n} = \sum \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n$ dvg (osc.)

26. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) = \frac{2}{3-x}$$

about the center $x_0 = 0$ and state when this Taylor series is valid. Hint:

$$f(x) = \frac{2}{3-x} = \left(\frac{2}{3}\right)\left(\frac{1}{1-\frac{x}{3}}\right).$$

by simple algebra.

26soln. We know the Geometric Series (a Commonly Used Taylor Series): $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$, valid for $|r| < 1$. So

$$f(x) = \frac{2}{3-x} = \left(\frac{2}{3}\right)\left[\frac{1}{1-\frac{x}{3}}\right] \overset{\text{GS}}{=} \left(\frac{2}{3}\right)\left[\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n\right] = \sum_{n=0}^{\infty}\left(\frac{2}{3^1}\right)\left(\frac{x^n}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{2}{3^{n+1}}\right)x^n.$$

And GS is valid $\iff \left|\frac{x}{3}\right| < 1 \iff |x| < 3$.

27. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) = \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$. Hint. Start with the Taylor series expansion

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{valid for } |x| < 1$$

and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} = (-1)(1-x)^{-2}\, D_x(1-x) = (-1)(1-x)^{-2}(-1) = (1-x)^{-2}.$$

Answer:

27soln.

$\boxed{2.}$ Start with Geometric Series and take Derivatives as many times as need. Geometric Series is valid when $|x|<1$ so resulting power series expansions will also be valid when $|x|<1$.

Geometric Series $\Rightarrow$ $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \overset{D_x}{\Longrightarrow} (1-x)^{-2} = \sum_{k=1}^{\infty} k\, x^{k-1}$

$\overset{D_x}{\Longrightarrow} 2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1)\, x^{k-2} \overset{D_x}{\Longrightarrow} 2\cdot 3(1-x)^{-4} = \sum_{k=3}^{\infty} k(k-1)(k-2)\, x^{k-3}$

So

$$(1-x)^{-4} = \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{6}\, x^{k-3} = \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6}\, x^n$$

$\boxed{\text{let } k-3=n} \Rightarrow k = n+3$

28. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} c_n (x-x_0)^n$, which is valid in some interval $(x_0 - R, x_0 + R)$ with $R > 0$. What, if anything, can you say about the second derivative $f''(x_0)$ of f evaluated at x_0?

28soln. If a function can be represented by a power series (on some interval centered at x_0, with an non-zero radius of convergence), then that power series must be the Taylor series centered at x_0. So $c_n = \dfrac{f^{(n)}(x_0)}{n!}$ for each integer $n \geq 0$. So $c_2 = \dfrac{f^{(2)}(x_0)}{2!}$, i.e., $f^{(2)}(x_0) = (2!)\, c_2 = 2c_2$.

29. Find the 2nd order Taylor polynomial for the function $f(x) = \sqrt[3]{x}$ about the center $x_0 = 8$.

29soln. The 2nd order Taylor polynomial is given by $P_2(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \dfrac{f^{(2)}(x_0)}{2!}(x - x_0)^2$. Using the below Helpful Table we get $P_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{9(2^5)}(x-8)^2$.

we were given $x_0 = 8$			
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$\frac{f^{(n)}(x_0)}{n!}$
0	$x^{1/3}$	$8^{1/3} = 2$	$\frac{2}{0!} = \frac{2}{1} = 2$
1	$\frac{1}{3}x^{\frac{-2}{3}}$	$\frac{1}{3}\left(8^{\frac{1}{3}}\right)^{-2} = \frac{1}{3}\frac{1}{2^2} = \frac{1}{12}$	$\frac{\frac{1}{12}}{1!} = \frac{1}{12}$
2	$\frac{-2}{9}x^{\frac{-5}{3}}$	$\frac{-2}{9}\left(8^{\frac{1}{3}}\right)^{-5} = \frac{-2}{9}\frac{1}{2^5} = \frac{-2}{9(2^5)}$	$\frac{1}{2!}\frac{-2}{9(2^5)} = \frac{-1}{9(2^5)}$

30. Consider the function

$$f(x) = e^{-x} .$$

The 5th order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_5(x) \;=\; \sum_{n=0}^{5} \frac{(-x)^n}{n!} \;=\; 1 \;-\; x \;+\; \frac{x^2}{2!} \;-\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;-\; \frac{x^5}{5!} \;.$$

The 5th order Remainder term $R_5(x)$ is defined by $R_5(x) = f(x) - P_5(x)$ and so $e^{-x} \approx P_5(x)$ where the approximation is within an error of $|R_5(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_5(x)|$ that is valid for each $x \in (-1, 3)$.

30soln.

For each $x \in (-1,3)$, there exists $c \in (-1,3)$ so that

$$|R_5(x)| = \left| \frac{f^{(6)}(c)}{6!} (x-0)^6 \right| = \frac{1}{6!} e^{-c} |x|^6 \leq \frac{1}{6!} e^{-(-1)} 3^6$$