MARK BOX		
PROBLEM	POINTS	
0	24	
1-14	56=14x4	
15	10	
16	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
  You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, give it to Prof. Girardi to hold for you during the exam (and it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- Cheating is grounds for a F in the course.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.7–10.10 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

0. Fill-in the boxes.

0A. Power Series Consider a (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \,(x - x_0)^n \;, \tag{1}$$

with radius of convergence $R \in [0, \infty]$. (Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.) Without any other further information of $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

0A.1. The choices for the next 4 boxes are: **AC**, **CC**, **DIVG**, or **anything**.

AC stands for: *is always absolutely convergent.*
CC stands for: *is always conditionally convergent.*
DIVG stands for: *is always divergent.*
anything stands for: *can do anything* i.e., there are examples showing that it can AC, CC, or DIVG.

(1) At the center $x = x_0$, the power series in (1) $\boxed{\text{AC}}$.

(2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series in (1) $\boxed{\text{AC}}$.

(3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$, the power series in (1) $\boxed{\text{DIVG}}$.

(4) If $0 < R < \infty$, then for the endpoints $x = x_0 \pm R$, the power series in (1) $\boxed{\text{anything}}$.

0A.2. Consider the function $y = h(x)$ defined by the power series in (1) and assume $R > 0$.

(1) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

$$h'(x) \;=\; \sum_{n=\mathbf{1}}^{\infty} \boxed{n\, a_n\, (x - x_0)^{n-1}} \;. \tag{2}$$

What can you say about the radius of convergence of the power series in (2)?

$\boxed{\text{The power series in (2) has the same raduis of convergence as the power series in (1).}}$

(2) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\, dx \;=\; \sum_{n=0}^{\infty} \boxed{\frac{a_n}{n+1}\,(x - x_0)^{n+1}}\,\Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

0B. Taylor/Maclaurin Polynomials and Series.

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.
Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

0B.1. The formula for c_n is

$$c_n \;=\; \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

0B.2. In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) = \boxed{f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N}$$

0B.3. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n}$$

0B.4. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)}}$$ for some c between $\boxed{x}$ and $\boxed{x_0}$.

0B.5. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.

0C. Commonly Used Taylor Series. The first two are done for you (hint: you might need it later). The other choices for the below functions below are: e^x, $\cos x$, $\sin x$, and $\ln(1+x)$.

►. A power series representation for:

0C.1. the function $y = \boxed{\frac{1}{1-x}}$ is $\sum_{n=0}^{\infty} x^n$, valid when $\boxed{x \in (-1,1)}$.

0C.2. the function $y = \boxed{\arctan x}$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, valid when $\boxed{x \in [-1,1]}$.

0C.3. the function $y = \boxed{\ln(1+x)}$ is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, valid when $\boxed{x \in (-1,1]}$.

0C.4. the function $y = \boxed{\cos x}$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, valid when $\boxed{x \in \mathbb{R}}$.

0C.5. the function $y = \boxed{\sin x}$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, valid when $\boxed{x \in \mathbb{R}}$.

0C.6. the function $y = \boxed{e^x}$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, valid when $\boxed{x \in \mathbb{R}}$.

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 4 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 1 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
14	14a	14b	14c	14d	14e					

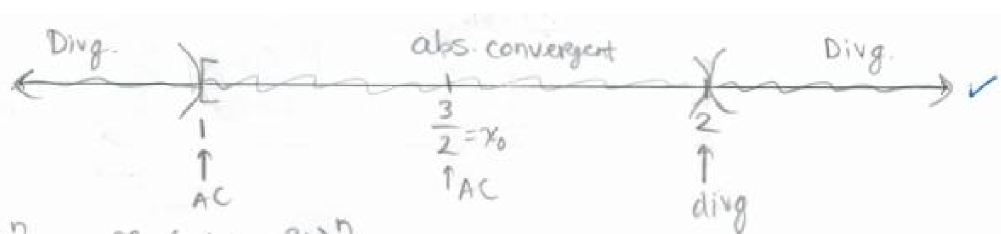
4	1	0	0
Total:			

15. Consider the formal power series

$$\sum_{n=2}^{\infty} \frac{(2x-3)^n}{n}.$$

The center is $x_0 = \underline{\quad \frac{3}{2} \quad}$ and the radius of convergence is $R = \underline{\quad \frac{1}{2} \quad}$.
As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.

15soln.



$$\sum_{n=2}^{\infty} \frac{(2x-3)^n}{n} = \sum_{n=2}^{\infty} \frac{\left(2\left(x-\frac{3}{2}\right)\right)^n}{n}, \quad x_0 = \frac{3}{2}$$

Use Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\left(2\left(x-\frac{3}{2}\right)\right)^{n+1}}{n+1} \cdot \frac{n}{\left(2\left(x-\frac{3}{2}\right)\right)^n} \right| = \lim_{n\to\infty} \left| \frac{\left(2\left(x-\frac{3}{2}\right)\right) \cdot n}{n+1} \right| =$$

$$= \left|2\left(x-\tfrac{3}{2}\right)\right| \lim_{n\to\infty} \frac{n}{n+1} = \left|2\left(x-\tfrac{3}{2}\right)\right| \cdot 1 = \left|2\left(x-\tfrac{3}{2}\right)\right| < 1$$

same degree

$$\left|x-\tfrac{3}{2}\right| < \tfrac{1}{2}$$

Radius of convergence $= \frac{1}{2}$

endpts $\Rightarrow$ $x=1$ and $x=2$

$x=1$:

$$\sum_{n=2}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n}$$

c_n

$c_n \geq 0$, $c_n > c_{n+1}$, $\lim_{n\to\infty} \frac{1}{n} = 0$ } so convergent by AST

$x=2$:

$$\sum_{n=2}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=2}^{\infty} \frac{1^n}{n} = \sum_{n=2}^{\infty} \frac{1}{n}$$, p-series, $p=1$ so divg.

We Did This Problem in Lectures.

16. Justify your answers. Show all your work.

16.1. Using the facts (so you do NOT need to show these facts) that $\arctan 1 = \frac{\pi}{4}$ and that

$$\arctan x \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{valid for} \quad x \in [-1,1], \tag{16.1}$$

express the number π as an alternating series $\sum_{n=0}^{\infty} a_n$.

answer: $\pi = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$

16soln.

$\tan^{-1}(1) = \frac{\pi}{4} \qquad 4\tan^{-1}(1) = \pi$

$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $x \in [-1, 1]$

$\pi = 4\tan^{-1}(1) = 4 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$

16.2. In Part **16.1** you found a_n's so that $\pi = \sum_{n=0}^{\infty} a_n$. Now estimate the error in approximating π by the partial sums $\sum_{n=0}^{N} a_n$ of your infinite series $\sum_{n=0}^{\infty} a_n$ in Part **16.1** . You may use (choice is yours) Taylor Remainder Theorem or Alternating Series Remainder Estimate. (You answer will have an N in it).

answer: $\left| \pi - \sum_{n=0}^{N} a_n \right| \leq \frac{4}{2(N+1)+1} \overset{\text{or}}{=} \frac{4}{2N+3}$

16soln.

the Taylor Reminder Theorem or the Alternating Series Reminder Estimate. ← yes!

b/c $0 \leq \frac{4}{2n+1} \searrow 0$ so

answer: $\left| \pi - \sum_{n=0}^{N} a_n \right| \leq \frac{4}{2N+3}$

$\left| \pi - \sum_{n=0}^{N} a_n \right| \leq \left| a_{N+1} \right| = \left| \frac{(-1)^{N+1} 4}{2(N+1)+1} \right| = \frac{4}{2N+3}$

STATEMENT OF MULTIPLE CHOICE PROBLEMS

These sheets of paper are <u>not</u> collected.

1. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} c_n x^n$, which is valid in some interval J containing 0 and that the raduis of J striclty positive.

1soln. If a function can be represented by a power series (on some interval centered at x_0, with an non-zero radius of convergence), then that power series must be the Taylor series centered at x_0. So $c_0 = \dfrac{f^{(0)}(x_0)}{0!} = f(0)$.

2. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} a_n x^n$, which is valid in some interval J containing 0 and the raduis of J striclty positive. Consider the two statements:
(A) If $y = f(x)$ is an even function (i.e., $f(-x) = f(x)$), then $a_1 = a_3 = a_5 = \cdots = 0$.
(B) If $y = f(x)$ is an odd function (i.e., $f(-x) = -f(x)$), then $a_0 = a_2 = a_4 = \cdots = 0$.

2soln. Both (A) and (B) are true.

*2.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-R, R)$.

a. Show that if f is even, then $a_1 = a_3 = a_5 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only even powers of x.

b. Show that if f is odd, then $a_0 = a_2 = a_4 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only odd powers of x.

It is known that all power series that converge to a function f(x) on an interval $(-R,R)$ are the same. This is a key property of power series that will be needed to complete this proof.

a. If f(x) is even, then $f(-x) =$ (1) __________.

Substitute $-x$ for x in the series $\sum_{n=0}^{\infty} a_n x^n$. What are the coefficients of the resulting power series for odd n?

The coefficients for odd n are (2) __________.

How does this show that the Taylor series for an even function f at $x = 0$ contains only even powers of x?

- ◉ **A. The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $-a_n$. The only solution to $a_n = -a_n$ is $a_n = 0$.**
- ○ B. The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $2a_n$. The only solution to $a_n = 2a_n$ is $a_n = 0$.
- ○ C. The substitution of $-x$ resulted in a coefficient of 0 for all odd n, so the statement has been proven.
- ○ D. The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $\frac{1}{2}a_n$. The only solution to $a_n = \frac{1}{2}a_n$ is $a_n = 0$.

b. If f(x) is odd, then $f(-x) =$ (3) __________.

Substitute $-x$ for x in the series $\sum_{n=0}^{\infty} a_n x^n$. What are the coefficients of the resulting power series for even n?

The coefficients for even n are (4) __________.

How does this show that the Taylor series for an odd function f at $x = 0$ contains only odd powers of x?

- ○ A. The substitution of $-x$ resulted in a coefficient of 0 for all even n, so the statement has been proven.
- ○ B. The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $\frac{1}{2}a_n$. The only solution to $a_n = \frac{1}{2}a_n$ is $a_n = 0$.
- ○ C. The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $2a_n$. The only solution to $a_n = 2a_n$ is $a_n = 0$.
- ◉ **D. The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $-a_n$. The only solution to $a_n = -a_n$ is $a_n = 0$.**

(1) ◉ **f(x)** ○ $-f(x)$

(2) ○ a_n ◉ $\mathbf{-a_n}$ ○ 0 ○ $2a_n$ ○ $\frac{1}{2}a_n$

(3) ○ f(x) ◉ $\mathbf{-f(x)}$

(4) ◉ $\mathbf{a_n}$ ○ $-a_n$ ○ 0 ○ n ○ $\frac{1}{2}a_n$

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►. Problems **1** and **2** were meant to help you with Problem **0C**. ☺

3. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{10^n}$$

Recall that the interval of convergence is the set of x's for which the power series converges, either absolutely or conditionally.

3soln. The interval of convergence is $(-8, 12)$.

$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim\limits_{n\to\infty}\left|\frac{(x-2)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(x-2)^n}\right|<1 \Rightarrow \frac{|x-2|}{10}<1 \Rightarrow |x-2|<10 \Rightarrow -10<x-2<10 \Rightarrow -8<x<12$; when $x=-8$ we have $\sum\limits_{n=1}^{\infty}(-1)^n$, a divergent series; when $x=12$ we have $\sum\limits_{n=1}^{\infty}1$, a divergent series

(a) the radius is 10; the interval of convergence is $-8<x<12$
(b) the interval of absolute convergence is $-8<x<12$
(c) there are no values for which the series converges conditionally

4. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, x^n}{n!}$$

Recall that the interval of convergence is the set of x's for which the power series converges, either absolutely or conditionally.

4soln. The interval of convergence is $(-\infty, \infty)$.

$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim\limits_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|<1 \Rightarrow |x|\lim\limits_{n\to\infty}\left(\frac{1}{n+1}\right)<1$ for all x

(a) the radius is ∞; the series converges for all x
(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

5. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} n^n\, x^n$$

Recall that the interval of convergence is the set of x's for which the power series converges, either absolutely or conditionally.

5soln. The interval of convergence is $[0, 0]$, i.e., converges only when $x = 0$.
The root test is easier than the ratio test on this one.

$$\rho = \lim_{n\to\infty} |n^n x^n|^{\frac{1}{n}} = \lim_{n\to\infty} n\,|x| = \begin{cases} |x| \lim\limits_{n\to\infty} n = \infty & \text{if } x \neq 0 \\ \lim\limits_{n\to\infty} n\,|0| = \lim\limits_{n\to\infty} 0 = 0 & \text{if } x = 0. \end{cases}$$

So if $x = 0$, then $\rho = 0 < 1$. And if $x \neq 0$, then $\rho = \infty > 1$.
If you want to use the ratio test:

$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim\limits_{n\to\infty}\left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right|<1 \Rightarrow |x|\left(\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)\left(\lim\limits_{n\to\infty}(n+1)\right)<1 \Rightarrow e|x|\lim\limits_{n\to\infty}(n+1)<1 \Rightarrow$ only $x=0$ satisfies this inequality

(a) the radius is 0; the series converges only for $x=0$
(b) the series converges absolutely only for $x=0$
(c) there are no values for which the series converges conditionally

6. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 3^{2n}\,(x-2)^n}{3n}$$

Recall that the interval of convergence is the set of x's for which the power series converges, either absolutely or conditionally.

6soln. The interval of convergence is $\left(\frac{17}{9}, \frac{19}{9}\right]$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{2n+2}(x-2)^{n+1}}{3(n+1)}\cdot\frac{3n}{(-1)^n 3^{2n}(x-2)^n}\right|<1 \Rightarrow |x-2|\lim_{n\to\infty}\frac{9n}{n+1}=9|x-2|<1 \Rightarrow \frac{17}{9}<x<\frac{19}{9};$$

when $x=\frac{17}{9}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}}{3n}\left(-\frac{1}{9}\right)^n=\sum_{n=1}^{\infty}\frac{1}{3n}$, a divergent series; when $x=\frac{19}{9}$ we have

$\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}}{3n}\left(\frac{1}{9}\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{3n}$, a conditionally convergent series.

(a) the radius is $\frac{1}{9}$; the interval of convergence is $\frac{17}{9}<x\le\frac{19}{9}$

(b) the interval of absolute convergence is $\frac{17}{9}<x<\frac{19}{9}$

(c) the series converges conditionally at $x=\frac{19}{9}$

7. A power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence R with $0 < R < \infty$ and is conditionally convergent at $x = x_1 \in \mathbb{R}$. What can you conclude about x_1?

7soln. Answer: x_1 must be either R or $-R$.
Recall, a power series can be conditionally convergent only at the endpoints.

8. Find the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$.

8soln. The computations below show that the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$ is $P_3(x) = 5 + 3(x-1) + 9(x-1)^2 + 10(x-1)^3$.

we were given $x_0 = 1$			
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$\frac{f^{(n)}(x_0)}{n!}$
0	$x^5 - x^2 + 5$	5	$\frac{5}{0!} = \frac{5}{1} = 5$
1	$5x^4 - 2x$	$5 - 2 = 3$	$\frac{3}{1!} = \frac{3}{1} = 3$
2	$5 \cdot 4x^3 - 2$	$20 - 2 = 18$	$\frac{18}{2!} = \frac{18}{2} = 9$
3	$5 \cdot 4 \cdot 3x^2$	$(5)(4)(3)$	$\frac{(5)(4)(3)}{3!} = \frac{(5)(4)(3)}{(3)(2)} = \frac{(5)(4)}{2} = 10$

9. Find the 17th order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = 6x^7 + 5$.

9soln. Note that the n^{th}-derivative of a polynomial of degree N is zero if $n > N$. So if $n \geq 8$, then $f^{(n)}(x) = 0$ for all $x \in \mathbb{R}$. Let's follow the notation from Problem **0B**.
If $N \geq 7$, then $f^{(N+1)}(c) = 0$ for any $c \in \mathbb{R}$ and so

$$|R_N(x)| = \left|\frac{f^{(N+1)}(c)}{(N+1)!}(x-1)^{N+1}\right| = \frac{0}{(N+1)!}|x-1|^{N+1} = 0 ,$$

and so $P_N(x) = f(x)$. So $P_{17}(x) = f(x)$.

10. Using the geometric series, find a power series representation about (i.e., centered at) $x = 5$ for the function

$$g(x) = \frac{3}{x-2}$$

and indicate when the representation is valid.

10soln. Answer: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n (x-5)^n$, valid on $(2, 8)$.

Use the Geometric Series, i.e., $\displaystyle\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ when $|r| < 1$. Note that $\left|\frac{x-5}{3}\right| < 1 \Leftrightarrow |x-5| < 3 \Leftrightarrow$ [the distance between x is 5 is striclty less than 3] $\Leftrightarrow x \in (5-3, 5+3) \Leftrightarrow x \in (2, 8)$.

$$g(x) = \frac{3}{x-2} = \frac{3}{3-[-(x-5)]} = \frac{1}{1-\left[-\left(\frac{x-5}{3}\right)\right]} = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n (x-5)^n,$$ which converges for

$\left|\frac{x-5}{3}\right| < 1$ or $2 < x < 8$.

11. Do the interval and radius of convergence of a power series change when the series is differentiated or integrated? Explain.

11soln. Answer: The interval may (or may not) change, but the radius of convergence does not change. We learned that the radius always remains the same but *there might be trouble at the endpoint.* I.e., The interval may change (due to the endpoints), but the radius of convergence does not change.

12. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) \;=\; \frac{x^2}{(1-x)^3}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$.

Hint. Notice $\frac{x^2}{(1-x)^3} = (x-0)^2 \left[\frac{1}{(1-x)^3}\right]$ and so a power series representation for $\frac{1}{(1-x)^3}$ centered about $x_0 = 0$ would be a helpful start. We know

$$(1-x)^{-1} \overset{\text{i.e.}}{=} \frac{1}{1-x} \underset{\text{Series}}{\overset{\text{Geometric}}{=}} \sum_{k=0}^{\infty} x^k \qquad \text{, valid for } |x| < 1 \,. \tag{GS}$$

Differentiating the left hand side of (GS) we get

$$D_x(1-x)^{-1} \;=\; (-1)(1-x)^{-2}\,D_x(1-x) \;=\; (-1)(1-x)^{-2}\,(-1) \;=\; (1-x)^{-2} \;=\; \frac{1}{(1-x)^2}\,.$$

Keep going.

12soln. Answer: $\displaystyle\sum_{n=2}^{\infty} \frac{(n)(n-1)}{2}\, x^n$.

Starting with the Geometric Series and differentiating we get the following power series expansions (center at $x_0 = 1$ and with radius of convergence 1),

$$\begin{aligned}
\frac{1}{1-x} \;&=\; \sum_{k=0}^{\infty} x^k \\
D_x\left((1-x)^{-1}\right) \;=\; (1-x)^{-2} \;&=\; \sum_{k=1}^{\infty} k x^{k-1} \\
D_x\left((1-x)^{-2}\right) \;=\; 2\,(1-x)^{-3} \;&=\; \sum_{k=2}^{\infty} k\,(k-1)\,x^{k-2}
\end{aligned}$$

and so

$$\frac{1}{(1-x)^3} \;=\; \sum_{k=2}^{\infty} \frac{k\,(k-1)}{2} x^{k-2}$$

So we get the following power series expansion (center at $x_0 = 1$, with radius of convergence 1):

$$\frac{x^2}{(1-x)^3} \;=\; x^2\left[\frac{1}{(1-x)^3}\right] \;=\; x^2\left[\sum_{k=2}^{\infty} \frac{k\,(k-1)}{2} x^{k-2}\right] \;=\; \sum_{k=2}^{\infty} \frac{k\,(k-1)}{2} x^{k-2} x^2 \;=\; \sum_{k=2}^{\infty} \frac{k\,(k-1)}{2} x^k.$$

13. Using a known (commonly used) Taylor series, express

$$\int \arctan(t^2)\,dt$$

as a power series (valid for $|t| < 1$).

13soln. Answer: $C + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+3}}{(2n+1)(4n+3)}$.

The *Commonly Used Taylor Series* $\arctan x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ is valid when $-1 \le x \le 1$. So

$$\arctan\left(t^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(t^2\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{2n+1}\ ,$$

which is valid when $-1 \le t^2 \le 1$, so valid when $-1 \le t \le 1$. By integrating the above power series expansion

$$\int \arctan\left(t^2\right)\, dt = \int \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{2n+1}\right] dt = \sum_{n=0}^{\infty} \left[\int (-1)^n \frac{t^{4n+2}}{2n+1}\, dt\right]$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+3}}{(2n+1)\,(4n+3)}\ ,$$

which is valid when (might loose the endpoints) $-1 < t < 1$.

14. Consider the function

$$f(x) = e^x\ .$$

The 4^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_4(x) \ = \ \sum_{n=0}^{4} \frac{x^n}{n!} \ = \ 1 \ + \ x \ + \ \frac{x^2}{2!} \ + \ \frac{x^3}{3!} \ + \ \frac{x^4}{4!}\ .$$

The 4^{th} order Remainder term $R_4(x)$ is defined by $R_4(x) = f(x) - P_4(x)$ and so $e^x \approx P_4(x)$ where the approximation is within an error of $|R_4(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_4(x)|$ that is valid for each $x \in (-8, 3)$.

14soln. Answer: $\dfrac{(e^3)(8^5)}{5!}$

By Taylor's Remainder Theorem, for each $x \in (-8, 3)$, there exists c between x and 0 so that

$$R_4\,(x) = \frac{f^{(5)}\,(c)\ \ (x-0)^5}{5!}\ .$$

Note that if $x \in (-8, 3)$ and c is between x and 0, then $c \in (-8, 3)$. Also, $\left|f^{(5)}\,(x)\right| = e^x$. So for each $x \in (-8, 3)$,

$$|R_4\,(x)| = \left|\frac{f^{(5)}\,(c)\ \ (x-0)^5}{5!}\right| = \frac{\left|f^{(5)}\,(c)\right|\ \ |x|^5}{5!} = \frac{e^c\ \ |x|^5}{5!} \le \frac{e^c\ 8^5}{5!} \le \frac{e^3\ 8^5}{5!}\ .$$