MARK BOX		
PROBLEM	POINTS	
0	10	
1	14	
2-15	56=14x4	
16	10	
17	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
  You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, give it to Prof. Girardi to hold for you during the exam (and it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- Cheating is grounds for a F in the course.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.1–10.6 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0.1. For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums, so $s_n = \sum_{k=1}^{n} a_k$. By definition, the formal series $\sum a_n$ converges if and only if

the $\lim_{n\to\infty} s_n$ converges (to a finite number). [also ok: the $\lim_{n\to\infty} s_n$ exists (in $\mathbb{R}$)]

0.2. ***p*-series**. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if $\boxed{p > 1}$.

0.3. **Geometric Series**. Fill in the boxes with the proper range of $r \in \mathbb{R}$.

- The series $\sum r^n$ converges if and only if r satisfies $\boxed{|r| < 1}$.

0.4. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If $\boxed{0 \le a_n \le c_n}$ (only $a_n \le c_n$ is also ok b/c given $a_n \ge 0$) when $n \ge 17$ and $\sum c_n$ converges, then $\sum a_n$ converges.
- If $\boxed{0 \le d_n \le a_n}$ (need $0 \le d_n$ part here) when $n \ge 17$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.5. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $\boxed{0 < L < \infty}$, then [$\sum b_n$ converges $\iff$ $\sum a_n$ converges].
- If $L = 0$, then $\boxed{[\ \sum b_n \text{ converges} \implies \sum a_n \text{ converges}\ ]}$.
- If $L = \infty$, then $\boxed{[\ \sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}\ ]}$.

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.6. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \ .$$

- If $\rho > 1$, then $\sum a_n$ $\boxed{\text{diverges}}$.
- If $\rho < 1$, then $\sum a_n$ $\boxed{\text{converges absolutely}}$.

1. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true.

The symbol $\sum$ is understood to mean $\sum_{n=1}^{\infty}$.

Scoring this problem: A problem with precisely one answer marked and the answer is correct, 1 point. All other cases, 0 points.

On the next 3: think of the n^{th}-term test for divergence and what if $a_n = \frac{1}{n}$.			
1.1	T		If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
1.2	T		If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
1.3		F	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
On the next 5: think of def. of AC/CC/Divergent, the Big Theorem AC $\Rightarrow$ convergence, and $\sum \frac{(-1)^n}{n}$ is CC.			
1.4	T		A series $\sum a_n$ is precisely one of the following: absolutely convergent, conditionally convergent, divergent.
1.5	T		If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ is either absolutely convergent or divergent.
1.6	T		If $\sum \lvert a_n \rvert$ converges, then $\sum a_n$ converges.
1.7	T		If $\sum a_n$ diverges, then $\sum \lvert a_n \rvert$ diverges.
1.8		F	If $\sum \lvert a_n \rvert$ diverges, then $\sum a_n$ diverges.
On the next 2: think of a Theorem from class and what if $f(x) = \sin(\pi x)$.			
1.9	T		If a function $f\colon [0,\infty) \to \mathbb{R}$ satisfies that $\lim_{x\to\infty} f(x) = L$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{n\to\infty} a_n = L$.
1.10		F	If a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies that $\lim_{n\to\infty} a_n = L$ and $f\colon [0,\infty) \to \mathbb{R}$ is a function satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{x\to\infty} f(x) = L$.
The next 4 are from a MML homework problem.			
1.11	T		The $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=17}^{\infty} a_n$ converges.
1.12		F	If $\sum a_n$ converges, then $\sum (a_n + 0.01)$ converges.
1.13		F	If $\sum_{n=1}^{\infty} p^n$ diverges, then $\sum_{n=1}^{\infty} (p+0.01)^n$ diverges, for a fixed real number p.
1.14		F	If $\sum_{n=1}^{\infty} n^{-p}$ converges, then $\sum_{n=1}^{\infty} n^{-p+0.01}$ converges, for a fixed real number p.

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 4 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 1 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
14	14a	14b	14c	14d	14e					
15	15a	15b	15c	15d	15e					
							4	1	0	0
							Total:			

16. Below the choice-boxes (AC/CC/Divg), carefully justify the behavior of the series. Be sure to clearly explain your logic and specify which test(s) you are using. Then check the correct choice-box.

$$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$$

- [ ] absolutely convergent
- [X] conditionally convergent
- [ ] divergent

16soln.

$\sum_{n=2}^{\infty} \left| (-1)^n \frac{\ln n}{n^{1/2}} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n^{1/2}}$ $\quad a_n = \frac{\ln n}{n^{1/2}} > \frac{1}{n^{1/2}} = b_n$

$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{\cancel{n^{1/2}}} \cdot \frac{\cancel{n^{1/2}}}{1} = \infty$ since $L = \infty$

if $\sum_{n=2}^{\infty} b_n$ diverges then $\sum_{n=2}^{\infty} a_n$ diverges.

$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges, P-series where $P = \frac{1}{2} \leq 1$

thus $\sum_{n=2}^{\infty} \left| (-1)^n \frac{\ln n}{n^{1/2}} \right|$ diverges by the Limit comparison test.

$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}} = \sum_{n=2}^{\infty} (-1)^n U_n$ where $U_n = \frac{\ln n}{\sqrt{n}}$

$U_n > 0$ for $n \geq 2$ $\quad \left(\frac{\ln n}{\sqrt{n}}\right)' = \frac{\sqrt{n}/n - \frac{\ln n}{2\sqrt{n}}}{n} = \frac{1/n^{1/2} - \frac{\ln n}{2n^{1/2}}}{n} =$

$U_{n+1} < U_n$ for a sufficiently large n $\quad \frac{\frac{2-\ln n}{2n^{1/2}}}{n} = \frac{2-\ln n}{2n^{3/2}} < 0$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \xrightarrow{} \frac{\infty}{\infty} = \lim_{n\to\infty} \frac{1/n}{\frac{1}{2\sqrt{n}}}$ $\quad 2 - \ln n < 0 \quad 2 < \ln n$ when $n > e^2$

$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$

$\lim_{n\to\infty} U_n = 0$ thus $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the alternating series test.

$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges and $\sum_{n=2}^{\infty} \left| (-1)^n \frac{\ln n}{\sqrt{n}} \right|$ diverges so

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$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ conditionall converges

17. Below the choice-boxes (AC/CC/Divg), carefully justify the behavior of the series. Be sure to clearly explain your logic and specify which test(s) you are using. Then check the correct choice-box. You may use Part 17.1 in Part 17.2.

17.1. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(3n-2)^n}$

- [X] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

17soln.

divergent

$\rho := \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{1}{(3n-2)^n}\right|} = \lim_{n\to\infty}\left[\left(\frac{1}{3n-2}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{1}{3n-2}$

$= 0$ $\quad \rho = 0 < 1$ thus $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(3n-2)^n}$

absolutely converges by the ratio test

17.2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(3n-2)^{n+\frac{1}{2}}}$

- [X] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

To clarify, the quantity $(3n-2)$ in the denominator is raised to the power $(n+\frac{1}{2})$.

17soln.

To clarify, the quantity $(3n-2)$ in the denominator is raised to the power $(n+\frac{1}{2})$.

$\sum_{n=1}^{\infty}\left|(-1)^n \frac{1}{(3n-2)^{n+1/2}}\right| = \sum_{n=1}^{\infty} \frac{1}{(3n-2)^{n+1/2}}$ $\quad a_n = \frac{1}{(3n-2)^{n+1/2}} \le \frac{1}{(3n-2)^{n}}$

$a_n \le c_n$ thus by DCT if $\sum_{n=1}^{\infty} c_n$ converges

then $\sum_{n=1}^{\infty} a_n$ converges $\quad \sum_{n=1}^{\infty} \frac{1}{(3n-2)^n}$ $\quad \rho := \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(3n-2)^n}\right|}$

$= \lim_{n\to\infty} \frac{1}{3n-2} = 0$ $\quad \rho = 0 < 1$ $\quad \sum c_n$ converges by the ratio test

thus $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{(3n-2)^{n+1/2}}\right|$ converges by DCT

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Since $\sum_{n=1}^{\infty}\left|(-1)^n \frac{1}{(3n-2)^n}\right|$ converges then $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(3n-2)^n}$ converges as well

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are <u>not</u> collected.

Abbreviations used:

- DCT is Direct Comparison Test.
- LCT is Limit Comparison Test.
- AST is Alternating Series Test.

2. Limit of a sequence. Evaluate

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}}\ .$$

2soln.

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2} \Rightarrow \text{ converges}$$

3. Limit of a sequence. Evaluate

$$\lim_{n\to\infty} \frac{\sin n}{n}.$$

3soln.

$\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

4. Consider the formal series

$$\sum_{n=1}^{\infty} \frac{1}{n+3^n}.$$

4soln.

$\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series $\left[|r| = \frac{1}{3} < 1\right]$, so $\sum\limits_{n=1}^{\infty} \frac{1}{n+3^n}$ converges by the Comparison Test.

5. Consider the formal series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}}\ .$$

5soln. $\frac{1}{\sqrt{(n+2)(n+7)}} \overset{n \text{ big}}{\sim} \frac{1}{\sqrt{(n)(n)}} = \frac{1}{n}$. So let $b_n = \frac{1}{n}$ and $a_n = \frac{(-1)^n}{\sqrt{(n+2)(n+7)}}$. Then

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{(n+2)(n+7)}} = \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{(n+2)(n+7)}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{(n+2)(n+7)}} = \sqrt{1} = 1$$

Since $0 < \lim_{n\to\infty} \frac{|a_n|}{b_n} < \infty$, by the LCT, $\sum b_n$ and $\sum |a_n|$ *do the same thing* and we know that $\sum b_n$ is the harmonic series so $\sum b_n$ is diverges. So $\sum |a_n|$ diverges.
Now let $u_n = \frac{1}{\sqrt{(n+2)(n+7)}}$. Since $0 \le u_n \searrow 0$, by the AST, $\sum (-1)^n u_n$ converges.
Now look at the choices.

6. Consider the formal series

$$\sum_{k=1}^{\infty} \frac{k^2}{e^k}.$$

6soln.

$\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}\right| = \lim_{k\to\infty} \left[\left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{e}\right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1,$$ so the series converges.

7. Consider the formal series

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}.$$

(Remember your Algebra Priorities: $e^{n^2} = e^{\left(n^2\right)}$.)

7soln.

Use the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1} n!} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1$

8. Consider the formal series

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right).$$

8soln.

Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

9. What is the smallest integer N such that the Alternating Series Estimate/Remainder Theorem guarentees that

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2}\right| \le 0.05?$$

Note that $0.05 = \frac{0.05}{1.0000} = \frac{5}{100} = \frac{1}{20}$.

9soln. Note that $0 \leq \frac{1}{n^2} \searrow 0$ so the AST applies and tells us that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges and that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \leq \frac{1}{(N+1)^2}.$$

So

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \overset{\text{have}}{\leq} \frac{1}{(N+1)^2} \overset{\text{want}}{\leq} \frac{1}{20}.$$

Note

$$\left[\frac{1}{(N+1)^2} \leq \frac{1}{20} \right] \Leftrightarrow \left[\, 20 \leq (N+1)^2 \, \right]$$

and $(3+1)^2 = 4^2 = 16$ and $(4+1)^2 = 5^2 = 25$.

10. Find the vaule for b for which

$$e^b + e^{2b} + e^{3b} + e^{4b} + \ldots = 9.$$

10soln. The geometric series $e^b + e^{2b} + e^{3b} + e^{4b} + \ldots = \sum_{n=1}^{\infty} e^{nb} = \sum_{n=1}^{\infty} \left(e^b\right)^n$ has ratio $r = e^b$ and so it converges precisely when $\left|e^b\right| < 1$. Note that

$$\left|e^b\right| < 1 \Longleftrightarrow 0 < e^b < 1 \Longleftrightarrow -\infty < b < 0.$$

So let

$$b < 0 \qquad \text{and so we know that} \qquad \sum_{n=1}^{\infty} \left(e^b\right)^n \qquad \text{converges.}$$

Let $s_n = \sum_{k=1}^{n} \left(e^b\right)^k$. Then

$$\begin{aligned} s_n &= e^b + e^{2b} + e^{3b} + \ldots + e^{(n-1)b} + e^{nb} \\ e^b s_n &= e^{2b} + e^{3b} + e^{4b} + \ldots + e^{nb} + e^{(n+1)b} \end{aligned}$$

$$\left(1 - e^b\right) s_n = e^b - e^{(n+1)b}$$

$$s_n \overset{\text{know } e^b \neq 1}{=} \frac{e^b - \left(e^b\right)^{n+1}}{1 - e^b} \xrightarrow{\text{as } n \to \infty, \text{ using that } |e^b| < 1} \frac{e^b}{1 - e^b}.$$

Note

$$9 = \frac{e^b}{1 - e^b} \Longleftrightarrow 9\left(1 - e^b\right) = e^b \Longleftrightarrow 9 - 9e^b = e^b \Longleftrightarrow 9 = 10e^b \Longleftrightarrow e^b = \frac{9}{10} \Longleftrightarrow b = \ln \frac{9}{10}.$$

11. Determine the behavior of the series

$$\sum_{n=1}^{\infty} \frac{2^n \, n! \, n!}{(2n)!}.$$

11soln.

converges by the Ratio Test: $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \frac{2^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n n! n!} = \lim\limits_{n\to\infty} \frac{2(n+1)(n+1)}{(2n+2)(2n+1)} = \lim\limits_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$

12. The series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1},$$

by the Direct Comparison Test,

12soln.

converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series

$n^2+1 > n^2 \Rightarrow n^2+1 > \sqrt{n}\cdot n^{3/2} \Rightarrow \frac{n^2+1}{\sqrt{n}} > n^{3/2} \Rightarrow \frac{\sqrt{n}}{n^2+1} < \frac{1}{n^{3/2}}$ or use Limit Comparison Test with $\frac{1}{n^{3/2}}$.

13. The series

$$\sum_{n=1}^{\infty} \frac{1}{1+\ln n},$$

by the Limit Comparison Test,

13soln.

diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{1+\ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{1+\ln n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} n = \infty$$

14. The series

$$\sum_{n=2}^{\infty} (-1)^{n+1}\, \frac{1}{n\ln n},$$

is

14soln.

converges conditionally since $f(x) = \frac{1}{x\ln x} \Rightarrow f'(x) = -\frac{[\ln(x)+1]}{(x\ln x)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n > u_{n+1} > 0$ for

$n \geq 2$ and $\lim_{n\to\infty} \frac{1}{n\ln n} = 0 \Rightarrow$ convergence; but by the Integral Test, $\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{b\to\infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x}\right) dx$

$= \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \lim_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n\ln n}$ diverges

15. Consider the formal series

$$\sum_{n=1}^{\infty} (-1)^n\, \frac{(2n+1)^n}{n^{2n}}.$$

15soln.

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2n+1)^n}{n^{2n}}\right|} = \lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty} \left(\frac{2}{n} + \frac{1}{n^2}\right) = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

converges by the Root Test.