MARK BOX		
PROBLEM	POINTS	
0	10	
1-13	65=13x5	
14	5	
15	10	
16	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- This exam has 3 types of problems: Problem 0, Multiple Choice, and Show All Your Work.
  – **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
  – **For Multiple Choice problems**, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
  – **For the Show All Your Work problems**, to receive credit you **MUST**:
    (1) **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
    (2) if a line/box is provided, then:
    — show you work BELOW the line/box
    — put your answer on/in the line/box
    (3) if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- Cheating is grounds for a F in the course.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1-8.5, 8.7, 8.8 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in the blanks.

0.1. $\arcsin(\,\text{-}\frac{1}{2}\,) =$ $\underline{\quad \frac{-\pi}{6} \quad}$ (Your answers should be an angle in **RADIANS**.)

0.2. Double-angle Formula. Your answer should involve trig functions of θ, and not of 2θ.

$\sin(2\theta) =$ $\underline{\quad 2\ \sin\theta\cos\theta \quad}$

0.3. $\int \frac{du}{u} \overset{u\neq 0}{=}$ $\underline{\quad \ln\ |u| \quad}$ + C

0.4. $\int u^n\, du \overset{n\neq -1}{=}$ $\underline{\quad \frac{u^{n+1}}{n+1} \quad}$ +C

0.5. $\int e^u\, du =$ $\underline{\quad e^u \quad}$ +C

0.6. $\int \sec^2 u\, du =$ $\underline{\quad \tan u \quad}$ $+C$

0.7. $\int \sec u\, du =$ $\underline{\quad \ln|\sec u + \tan u| \overset{or}{=} \text{-}\ln|\sec u - \tan u| \quad}$ $+C$

0.8. $\int \frac{1}{a^2+u^2}\, du \overset{a\geq 0}{=}$ $\underline{\quad \frac{1}{a}\ \tan^{\text{-}\mathbf{1}}\left(\frac{u}{a}\right) \quad}$ $+C$

0.9. Trig sub.: if the integrand involves $u^2 - a^2$, then one makes the substitution $u =$ $\underline{\quad a\sec\theta \quad}$

0.10. Integration by parts formula: $\int u\, dv =$ $\underline{\quad uv - \int v du \quad}$

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
							5	2	0	0
							Extra Credit:			

14. Evaluate $\int (1+\sqrt{x})^{123}\, dx$. Show your work below the box then put your answer in the box.

$$\int (1+\sqrt{x})^{123}\; dx \;=\; \frac{2}{125}\,(1+\sqrt{x})^{125} \;-\; \frac{2}{124}\,(1+\sqrt{x})^{124} \qquad + \text{C}$$

14soln. Inspired by 81 Integrals number 23: $\int_0^1 (1+\sqrt{x})^8\, dx$.

$u = 1 + \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\, dx$

$2\sqrt{x}\, du = dx$

$u = 1 + \sqrt{x}$

$u - 1 = \sqrt{x}$

$2(u-1)\, du = dx$

$$\int (1+\sqrt{x})^{123} dx = 2\int (u)^{123}(u-1)\, du$$

$$= 2\int (u^{124} - u^{123})\, du$$

$$= 2\left(\frac{u^{125}}{125}\right) - 2\left(\frac{u^{124}}{124}\right) + C$$

$$= 2\left(\frac{(1+\sqrt{x})^{125}}{125} - \frac{(1+\sqrt{x})^{124}}{124}\right) + c$$

15. Show your work below the box then put your answer in the box. You may use part (a) in part (b).

15a. Evaluate $\int \sin^2 x\, dx$.

$$\int \sin^2 x \; dx \;=\; \frac{x}{2} - \frac{1}{4}\sin 2x + C \quad \overset{\text{also ok}}{=} \quad \frac{x}{2} - \frac{1}{2}\sin x \cos x \quad + \text{C}$$

$$= \int \frac{1-\cos 2x}{2}\, dx = \int \tfrac{1}{2}\, dx - \int \tfrac{1}{2}\cos 2x\, dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2x\right) + C$$

$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$$

15b. Evaluate $\int x \, \sin^2 x\, dx$.

$$\int x \sin^2 x \; dx \;=\; \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + C \quad \overset{\text{also ok}}{=} \quad \frac{x^2}{4} - \frac{x \sin x \cos x}{2} + \frac{\sin^2 x}{4} \quad + \text{C}$$

Is number 17 from the 81 Integrals. First Key Idea for Parts on the blue sheet.

$u = x \qquad dv = \sin^2 x\, dx$

$du = dx \qquad v = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x$

$$= x\left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right) - \left[\int \left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right) dx\right]$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{4}x\sin 2x - \int \tfrac{1}{2}x\, dx + \int \tfrac{1}{4}\sin 2x\, dx$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{4}\sin 2x - \tfrac{1}{4}x^2 + \tfrac{1}{4}\left(-\tfrac{1}{2}\cos 2x\right)$$

$$= \tfrac{1}{4}x^2 - \tfrac{1}{4}\sin 2x - \tfrac{1}{8}\cos 2x + C$$

17. $\int x \sin^2 x\, dx \quad \left[\begin{array}{ll} u = x, & dv = \sin^2 x\, dx, \\ du = dx & v = \int \sin^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)\, dx = \frac{1}{2}x - \frac{1}{2}\sin x \cos x \end{array}\right]$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x \sin x \cos x - \int \left(\tfrac{1}{2}x - \tfrac{1}{2}\sin x \cos x\right) dx$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x \sin x \cos x - \tfrac{1}{4}x^2 + \tfrac{1}{4}\sin^2 x + C = \tfrac{1}{4}x^2 - \tfrac{1}{2}x \sin x \cos x + \tfrac{1}{4}\sin^2 x + C$$

Note: $\int \sin x \cos x\, dx = \int s\, ds = \frac{1}{2}s^2 + C$ [where $s = \sin x$, $ds = \cos x\, dx$].

A slightly different method is to write $\int x \sin^2 x\, dx = \int x \cdot \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}\int x\, dx - \frac{1}{2}\int x \cos 2x\, dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8}\cos 2x + C$.

16. Part 16a should help with part 16b. Show your work below the box then put your answer in the box.

16a. Evaluate $\int e^{\left(x^2\right)}\,(2x)\,dx$.

$$\boxed{\int e^{\left(x^2\right)}\,(2x)\;dx \;=\; e^{x^2} \qquad + \text{ C}}$$

$$2\int e^{x^2} x \;=\; \frac{2}{2}\int e^u\,du = e^u = e^{x^2}$$

$u = x^2$

$du = 2x\,dx,\quad \frac{1}{2}du = x\,dx$

16b. We cannot integrate the functions $y = e^{x^2}$ and $y = x^2\,e^{x^2}$ with techniques we have learned thus far (in fact, they do not have elementary antiderivatives). But we can integrate $y = (2x^2+1)\,e^{x^2}$ with techniques we know thus far. Evaluate $\int (2x^2+1)\,e^{x^2}\,dx$.

$$\boxed{\int \left(2x^2+1\right) e^{x^2}\;dx \;=\; xe^{x^2} \qquad + \text{ C}}$$

Is number 81 from the 81 Integrals.

$$\int (2x^2+1)e^{x^2}dx = \int 2x^2e^{x^2}dx + \int e^{x^2}dx$$

$$= \int x\,(2xe^{x^2})\,dx + \int e^{x^2}dx$$

$u = x \quad dv = 2xe^{x^2}dx$

$du = dx \quad v = e^{x^2}$

$$\int x\,(2xe^{x^2})\,dx = xe^{x^2} - \int e^{x^2}dx$$

$$\int (2x^2+1)e^{x^2}dx = xe^{x^2} - \int e^{x^2}dx + \int e^{x^2}dx$$

$$= xe^{x^2} + C$$

81. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int (2x^2+1)e^{x^2}\,dx = \int 2x^2e^{x^2}\,dx + \int e^{x^2}\,dx = \int x\left(2xe^{x^2}\right)dx + \int e^{x^2}\,dx$$

$$= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \quad \begin{bmatrix} u = x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{bmatrix} \quad = xe^{x^2} + C$$

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r\ln a$.

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1. Evaluate the integral

$$\int_0^1 \frac{x}{x^2+9}\,dx.$$

1soln.

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+9} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+9| + C$$

$$\boxed{\begin{aligned} u &= x^2+9 \\ du &= 2x\,dx \end{aligned}}$$

Check $D_x\ \frac{1}{2}\ln|x^2+9| = \frac{1}{2}\,\frac{1}{x^2+9}\cdot 2x = \frac{x}{x^2+9}$ ✓

So $\int_0^1 \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9|\Big|_{x=0}^{x=1} = \frac{1}{2}\ln 10 - \frac{1}{2}\ln 9$

$= \frac{1}{2}\left[\ln 10 - \ln 9\right] = \frac{1}{2}\ln\left(\frac{10}{9}\right)$

2. Evaluate the integral

$$\int_0^4 \frac{x}{x+9}\,dx\ .$$

2soln.

$$\int \frac{x}{x+9}\,dx = \int 1\,dx - 9\int \frac{dx}{x+9} = x - 9\ln|x+9| + C$$

$$\boxed{\frac{x}{x+9} = \frac{x+9}{x+9} - \frac{9}{x+9} = 1 - \frac{9}{x+9}}$$

Long Division (Fake)

Check $D_x\left[x - 9\ln|x+9|\right] = 1 - \frac{9}{x+9}$

$$= \frac{x+9}{x+9} - \frac{9}{x+9} = \frac{x}{x+9}\ . \checkmark$$

So $\int_0^4 \frac{x}{x+9}\,dx = \left[x - 9\ln|x+9|\right]\Big|_{x=0}^{x=4}$

$$= \left[4 - 9\ln|13|\right] - \left[0 - 9\ln|9|\right]$$

$$= 4 - 9\ln(13) + 9\ln(9)$$

3. Evaluate

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\,dx\ .$$

3soln.

$$\int \sin^2 x \cos^3 x\,dx = \int (\sin^2 x)(1-\sin^2 x)\cos x\,dx$$

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x\,dx \end{aligned}}$$

$$= \int u^2(1-u^2)\,du = \int (u^2 - u^4)\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \checkmark$$

$$\int_0^{\pi/2} \sin^2 x \cos^3 x\,dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\Bigg|_{x=0}^{x=\pi/2} = \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15} = \frac{2}{15} \checkmark$$

4. Evaluate

$$\int_{x=5}^{x=10} \frac{\sqrt{x^2-25}}{x}\, dx$$

AND specify the initial substitution.

4soln. The integrand has a $u^2 - a^2$, so we try $u = a\sec\theta$. $\boxed{x = 5\sec\theta \quad \text{so} \quad dx = 5\sec\theta\tan\theta\, d\theta}$

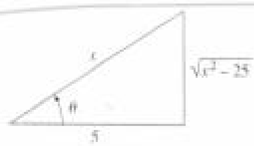
Thus,

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= 5\int \tan^2\theta\,d\theta \qquad \tan\theta \ge 0 \text{ since } 0 \le \theta < \pi/2$$

$$= 5\int(\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C$$



$x = 5\sec\theta$

Figure 8.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 8.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C \blacktriangleleft$$

• Check $D_x\left[(x^2-25)^{1/2} - 5\sec^{-1}\left(\frac{x}{5}\right)\right]$

$$= \frac{1}{2}(x^2-25)^{-1/2}(2x) - 5\frac{1}{\left|\frac{x}{5}\right|\sqrt{\left(\frac{x}{5}\right)^2-1}}\cdot\frac{1}{5}$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{1}{\frac{x}{5}\sqrt{\frac{x^2}{25}-\frac{25}{25}}} \qquad (\text{know } x \ge 5)$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{25}{x(x^2-25)^{1/2}} = \frac{(x^2-25)}{x(x^2-25)^{1/2}}\cdot\frac{(x^2-25)^{1/2}}{(x^2-25)^{1/2}}$$

$$= \frac{(x^2-25)(x^2-25)^{1/2}}{x(x^2-25)} = \frac{\sqrt{x^2-25}}{x}. \checkmark$$

• $$\int_5^{10}\frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)\Big|_{x=5}^{x=10}$$

$$= \left[\sqrt{100-25} - 5\sec^{-1}2\right] - \left[0 - \cancelto{0}{5\sec^{-1}1}\right] =$$

$$= \sqrt{75} - 5\cdot\frac{\pi}{3} = 5\sqrt{3} - 5\left(\frac{\pi}{3}\right) = 5\left(\sqrt{3} - \frac{\pi}{3}\right).$$

5. Evaluate the integral

$$\int_0^{\pi/2} \cos^3 x\, dx.$$

5soln. Break off one cos to form the $du = \cos x dx$ and so $u = \sin x$.

$$\int \cos^3 x\, dx = \int \left(\cos^2 x\right) \cos x\, dx = \int \left(1 - \sin^2 x\right) \cos x\, dx = \int \cos x\, dx - \int \sin^2 x \cos x\, dx = \sin x - \tfrac{1}{3}\sin^3 x + C$$

$$\begin{aligned} \int_0^{\pi/2} \cos^3 x\, dx &= \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} \\ &= \left[\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right] - \left[\sin 0 - \frac{1}{3}\sin^3 0\right] = \left[1 - \frac{1}{3}\right] - [0 - 0] = \frac{2}{3} \end{aligned}$$

6. Evaluate the integral

$$\int_1^2 \frac{8}{x^2 - 2x + 2}\, dx.$$

Hint: complete the square in the denominator.

6soln. $x^2 - 2x + 2 = (x-1)^2 + 1$. So $\frac{8}{x^2-2x+2} = \frac{8}{(x-1)^2+1}$. So let $u = x - 1$.

$$u = x - 1 \quad du = dx$$

$$u = 0 \text{ when } x = 1,\ u = 1 \text{ when } x = 2$$

$$\begin{aligned} \int_1^2 \frac{8}{x^2 - 2x + 2}\, dx &= 8\int_0^1 \frac{1}{u^2+1}\, du \\ &= 8\tan^{-1} u\Big]_0^1 = 8\left(\frac{\pi}{4} - 0\right) = 2\pi \end{aligned}$$

7. Evaluate the integral

$$\int_1^2 x \ln x\, dx.$$

7soln. $y = \ln x$ is easy to differentiate but hard to integrate so try parts with $u = \ln x$.

$$u = \ln x,\ du = \frac{dx}{x};\ dv = x\, dx,\ v = \frac{x^2}{2};$$

$$\int_1^2 x \ln x\, dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

8. Evaluate the integral

$$\int_0^{\pi/8} e^{3x} \cos(4x)\, dx.$$

8soln. First we show that

$$\int e^{3x} \cos 4x \; dx \;=\; \frac{e^{3x}}{25}\,(3\cos 4x + 4\sin 4x) + C\ .$$

We will use two integration by parts and the *bring to the other side* idea. For the <u>two</u> integration by parts, put the expontential function with either the u's <u>both</u> times or the dv's <u>both</u> times.

$\boxed{\text{Way \# 1}}$

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{3x} \qquad\qquad dv_1 = \cos 4x\, dx$$

$$du_1 = 3e^{3x}\, dx \qquad\qquad v_1 = \frac{1}{4}\sin 4x\ .$$

So by integration by parts

$$\int e^{3x} \cos 4x \; dx \;=\; \frac{1}{4}e^{3x}\sin 4x - \frac{3}{4}\int e^{3x}\sin 4x\, dx\ .$$

Now let

$$u_2 = e^{3x} \qquad\qquad dv_2 = \sin 4x\, dx$$

$$du_2 = 3e^{3x}\, dx \qquad\qquad v_2 = \frac{-1}{4}\cos 4x\ .$$

to get

$$\begin{aligned}\int e^{3x} \cos 4x \; dx \;&=\; \frac{1}{4}e^{3x}\sin 4x - \frac{3}{4}\left[\frac{-1}{4}e^{3x}\cos 4x - \frac{-3}{4}\int e^{3x}\cos 4x\, dx\right] \\ &=\; \frac{1}{4}e^{3x}\sin 4x + \frac{3}{4^2}e^{3x}\cos 4x - \frac{3^2}{4^2}\int e^{3x}\cos 4x\, dx\ .\end{aligned}$$

Now solving for $\int e^{3x}\cos 4x\, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{3^2}{4^2}\right]\int e^{3x} \cos 4x \; dx \;=\; \frac{1}{4}e^{3x}\sin 4x + \frac{3}{4^2}e^{3x}\cos 4x + K$$

and so

$$\begin{aligned}\int e^{3x} \cos 4x \; dx \;&=\; \left[\frac{4^2}{25}\right]\left(\frac{1}{4}e^{3x}\sin 4x + \frac{3}{4^2}e^{3x}\cos 4x + K\right) \\ &=\; \frac{4}{25}\, e^{3x}\sin 4x + \frac{3}{25}\, e^{3x}\cos 4x + \left[\frac{K4^2}{25}\right] \\ &=\; \frac{e^{3x}}{25}\,(4\sin 4x + 3\cos 4x) + \left[\frac{K4^2}{25}\right]\ .\end{aligned}$$

Thus

$$\int e^{3x} \cos 4x \; dx \;=\; \boxed{\frac{e^{3x}}{25}\,(3\cos 4x + 4\sin 4x) + C}\ .$$

$\boxed{\text{Way \# 2}}$

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$\begin{aligned} u_1 &= \cos 4x & dv_1 &= e^{3x} dx \\ du_1 &= -4 \sin 4x \, dx & v_1 &= \frac{1}{3} e^{3x} \; . \end{aligned}$$

So, by integration by parts

$$\int e^{3x} \cos 4x \; dx \;=\; \frac{1}{3} e^{3x} \cos 4x - \frac{-4}{3} \int e^{3x} \sin 4x \, dx \; .$$

Now let

$$\begin{aligned} u_2 &= \sin 4x & dv_2 &= e^{3x} dx \\ du_2 &= 4 \cos 4x \, dx & v_2 &= \frac{1}{3} e^{3x} \; . \end{aligned}$$

to get

$$\begin{aligned} \int e^{3x} \cos 4x \; dx \;&=\; \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \left[\frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \int e^{3x} \cos 4x \, dx \right] \\ &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3^2} e^{3x} \sin 4x - \frac{4^2}{3^2} \int e^{3x} \cos 4x \, dx \; . \end{aligned}$$

Now solving for $\int e^{3x} \cos 4x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{4^2}{3^2} \right] \int e^{3x} \cos 4x \; dx \;=\; \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3^2} e^{3x} \sin 4x + K$$

and so

$$\begin{aligned} \int e^{3x} \cos 4x \; dx \;&=\; \left[\frac{3^2}{3^2 + 4^2} \right] \left(\frac{1}{3} e^{3x} \cos 4x + \frac{4}{3^2} e^{3x} \sin 4x + K \right) \\ &= \frac{3}{25} \, e^{3x} \cos 4x + \frac{4}{25} \, e^{3x} \sin 4x + \left[\frac{K 3^2}{3^2 + 4^2} \right] \\ &= \frac{e^{3x}}{25} \, (3 \cos 4x + 4 \sin 4x) + \left[\frac{K 3^2}{3^2 + 4^2} \right] \end{aligned}$$

Thus

$$\int e^{3x} \cos 4x \; dx \;=\; \boxed{\frac{e^{3x}}{25} \, (3 \cos 4x + 4 \sin 4x) + C} \, .$$

$\boxed{\text{Doesn't Work Way}}$

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

$\boxed{\text{Now evaluate the indefinite integral.}}$

So

$$\begin{aligned}\int_0^{\pi/8} e^{3x} \cos(4x)\, dx &= \left[\frac{e^{3x}}{25}\,(3\cos 4x + 4\sin 4x)\right] \Big|_{x=0}^{x=\pi/8} \\ &= \left[\frac{e^{3\pi/8}}{25}\left(3\cos\frac{\pi}{2} + 4\sin\frac{\pi}{2}\right)\right] - \left[\frac{e^0}{25}\,(3\cos 0 + 4\sin 0)\right] \\ &= \left[\frac{e^{3\pi/8}}{25}\,(4)\right] - \left[\frac{1}{25}\,(3)\right] = \frac{4e^{3\pi/8} - 3}{25}\end{aligned}$$

9. Evaluate the integral

$$\int_0^{\sqrt{3}/2} \frac{4x^2\, dx}{(1-x^2)^{3/2}}$$

9soln.

$$x = \sin\theta,\ 0 \le \theta \le \tfrac{\pi}{3},\ dx = \cos\theta\, d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta;$$

$$\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{\left(1-x^2\right)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3}\left(\sec^2\theta - 1\right) d\theta = 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

10. Evaluate the integral

$$\int_0^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\, dt$$

10soln. Partial Fraction Decompostion. $t^3 + t = t\,(t^2+1) = (t-0)\,(t^2+1)$.

$$\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} = \frac{A\,(t^2+1) + (Bt+C)\,t}{t^3+t}.$$

So

$$3t^2 + t + 4 = A\left(t^2+1\right) + (Bt+C)\,t\ ,$$

which gives

$$\begin{aligned} t^2 &: \quad 3 = A + B \\ t^1 &: \quad 1 = \quad C \\ t^0 &: \quad 4 = \quad A \end{aligned}$$

and so $B = 1 - 3 = 3 - 4 = -1$. Thus

$$\int \frac{3t^2+t+4}{t^3+t}\, dt = \int \frac{4}{t}\, dt + \int \frac{-t+1}{t^2+1}\, dt\ .$$

Note

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{-t+1}{t^2+1}\, dt &= \frac{-1}{2}\int_0^{\sqrt{3}} \frac{2t\, dt}{t^2+1}\, dt + \int_0^{\sqrt{3}} \frac{1}{t^2+1}\, dt = \left[\frac{-1}{2}\ln\left|t^2+1\right| + \tan^{-1} t\right] \Big|_0^{\sqrt{3}} \\ &= \left[\frac{-\ln 4}{2} + \tan^{-1}\sqrt{3}\right] - \left[\frac{-\ln 1}{2} + \tan^{-1} 0\right] = \left[\frac{-\ln(2^2)}{2} + \frac{\pi}{3}\right] - [0-0] = \frac{\pi}{3} - \ln 2.\end{aligned}$$

Also

$$\int_0^{\sqrt{3}} \frac{4}{t}\,dt = 4 \lim_{b\to 0+} \int_b^{\sqrt{3}} \frac{dt}{t} = 4 \lim_{b\to 0+} \ln|t| \,\Big|_b^{\sqrt{3}} = 4 \lim_{b\to 0+} \left[\ln\sqrt{3} - \ln b\right] = \infty .$$

So $\int_0^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$ diverges to infinity.

11. Let $y = p(x)$ be a polynomial of degree 5.
What is the form of the partial fraction decomposition of

$$\frac{p(x)}{(x^2-1)\,(x^2+1)^2} \ ?$$

Here A, B, C, D, E and F are constants.

11soln. First note degree of den. $= 2+(2)(2) = 6 > 5 =$ dgree of num. (i.e., we have *strictly bigger bottoms*) and so we do not have to first do long division. So next factor the dem.: $(x^2-1)\,(x^2+1)^2 = (x-1)\,(x+1)\,(x^2+1)^2$ where $x-1$ and $x+1$ are linear terms while x^2+1 is an irreducible quadratic term. Now see the partial fraction handout from class to see that the PDF takes the form $\dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$.

12. Evaluate

$$\int_{-1}^{1} \frac{dx}{x^2} \, .$$

12soln. $\int_{-1}^{1} \frac{dx}{x^2}$ diverges to ∞.
This is an improper integral (the integrand $y = \frac{1}{x^2}$ is continuous on $(-\infty,0)\cup(0,\infty)$ but is not defined at $x=0$) and

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[\int_{-1}^{0} \frac{dx}{x^2}\right] + \left[\int_{0}^{1} \frac{dx}{x^2}\right] = \left[\lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^2}\right] + \left[\lim_{a\to 0^+} \int_{a}^{1} \frac{dx}{x^2}\right] .$$

Note

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x^2} = \lim_{a\to 0^+} \int_a^1 x^{-2}\,dx = \lim_{a\to 0^+} -x^{-1}\,\Big|_a^1 = \lim_{a\to 0^+} \left[(-1) - \left(\frac{-1}{a}\right)\right] = \lim_{a\to 0^+} \left[\frac{1}{a} - 1\right] = \infty .$$

The graph of $y = \frac{1}{x^2}$ is symmetric about the y axis and so $\lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^2} = \infty$; or, you can calculate this limit similarly to the limit we just calculated. So

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[\lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^2}\right] + \left[\lim_{a\to 0^+} \int_{a}^{1} \frac{dx}{x^2}\right] = [\infty] + [\infty] = \infty \, .$$

13. Evaluate

$$\int_{-1}^{1} \frac{dx}{x^3} \, .$$

13soln. $\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx$ does not exist but also does not diverge to infinity.
This is an improper integral (the integrand $y = \frac{1}{x^3}$ is continuous on $(-\infty,0)\cup(0,\infty)$ but is not defined at $x=0$) and

$$\int_{-1}^{1} \frac{dx}{x^3} = \left[\int_{-1}^{0} \frac{dx}{x^3}\right] + \left[\int_{0}^{1} \frac{dx}{x^3}\right] = \left[\lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^3}\right] + \left[\lim_{a\to 0^+} \int_{a}^{1} \frac{dx}{x^3}\right]$$

• $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$

$\int_{x=0}^{x=1} x^{-3}\,dx = \lim_{a\to 0^+} \frac{x^{-2}}{-2}\Big|_{x=a}^{x=1} = \frac{1}{2}\lim_{a\to 0^+}\left[\frac{1}{x^2}\right]_{x=1}^{x=a} =$

$\frac{1}{2}\lim_{x\to 0^+}\left[\frac{1}{a^2} - 1\right] = \infty$. Similarly, $\int_{-1}^{0} x^{-3}\,dx = -\infty$.

• $\int_{-1}^{1} x^{-3}\,dx = \int_{-1}^{0} x^{-3}\,dx + \int_{0}^{1} x^{-3}\,dx = -\infty + \infty$ so DNE.