MARK BOX		
PROBLEM		
1-30	30	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) NOT TO HAND-IN PART. This part will not be collected.
  Take this part home to learn from and to check your answers when the solutions are posted.
- **For the Multiple Choice** problems, circle your answer(s) on the provided chart.
  No need to show work.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold during the exam (it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets.
- Cheating is grounds for a F in the course.
- At a student's request, I will project my watch upon the projector screen.
  Upon request, you will be given as much (blank) scratch paper as you need.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET):
  §8.1–8.5, 8.7, 8.8, 10.1–10.10, 11.1–11.5 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table your solution to the multiple choice problems.
- $\boxed{\text{You may choose up to } \mathbf{1} \text{ answers for each multiple choice problem}}$.
  The scoring is as follows.
  * For a problem with precisely one answer marked and the answer is correct, $\frac{10}{3} = 3.\overline{3}$ points.
  * All other cases, 0 points.

At most ONE choice per problem. Table for Your Muliple Choice Solutions						
PROBLEM						leave this column blank
1	1a	1b	1c	1d	1e	
2	2a	2b	2c	2d	2e	
3	3a	3b	3c	3d	3e	
4	4a	4b	4c	4d	4e	
5	5a	5b	5c	5d	5e	
6	6a	6b	6c	6d	6e	
7	7a	7b	7c	7d	7e	
8	8a	8b	8c	8d	8e	
9	9a	9b	9c	9d	9e	
10	10a	10b	10c	10d	10e	
11	11a	11b	11c	11d	11e	
12	12a	12b	12c	12d	12e	
13	13a	13b	13c	13d	13e	
14	14a	14b	14c	14d	14e	
15	15a	15b	15c	15d	15e	
16	16a	16b	16c	16d	16e	
17	17a	17b	17c	17d	17e	
18	18a	18b	18c	18d	18e	
19	19a	19b	19c	19d	19e	
20	20a	20b	20c	20d	20e	
21	21a	21b	21c	21d	21e	
22	22a	22b	22c	22d	22e	
23	23a	23b	23c	23d	23e	
24	24a	24b	24c	24d	24e	
25	25a	25b	25c	25d	25e	
26	26a	26b	26c	26d	26e	
27	27a	27b	27c	27d	27e	
28	28a	28b	28c	28d	28e	
29	29a	29b	29c	29d	29e	
30	30a	30b	30c	30d	30e	17

NOT TO HAND-IN PART
STATEMENT OF MULTIPLE CHOICE PROBLEMS

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. Laws of Logs. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.
- Abbreviations used with Series:
  - DCT is Direct Comparison Test.
  - LCT is Limit Comparison Test.
  - AST is Alternating Series Test.

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1. Evaluate the integral

$$\int_0^1 \frac{x}{x^2+9}\,dx.$$

1soln.

$$\int_{x=0}^{x=1} \frac{x}{x^2+9}\,dx = \frac{1}{2}\int_{u=9}^{u=10} \frac{du}{u} = \frac{1}{2} \ln|u| \Big|_{u=9}^{u=10}$$

$$\boxed{\begin{array}{l} u = x^2+9 \\ du = 2x\,dx \\ \hline x=0 \Rightarrow u=9 \\ x=1 \Rightarrow u=10 \end{array}}$$

$$= \frac{1}{2}\ln 10 - \frac{1}{2}\ln 9$$

2. Evaluate the integral

$$\int_0^4 \frac{x}{x+9}\, dx \ .$$

2soln.

$$\int_0^4 \frac{x}{x+9}\, dx$$

Do not have strictly bigger bottoms so need to do long division.
But it's easy to "fake" long division here:

$$\frac{x}{x+9} = \frac{x+9-9}{x+9} = \frac{x+9}{x+9} - \frac{9}{x+9} = 1 - \frac{9}{x+9}.$$

So

$$\int_0^4 \frac{x}{x+9}\, dx = \int_0^4 \left[1 - \frac{9}{x+9}\right] dx$$

$$= \left[x - 9 \ln|x+9|\right] \Big|_{x=0}^{x=4}$$

$$= (4 - 9 \ln 13) - (0 - 9 \ln 9)$$

$$= 4 - 9 \ln 13 + 9 \ln 9 .$$

3. Evaluate

$$\int_0^{\ln(2\pi)} e^x \cos(e^x)\ dx$$

3soln. Let $u = e^x$. So $du = e^x dx$. So $\int e^x \cos(e^x)\ dx = \int \cos u\, du = \sin u + C = \sin(e^x) + C$.
Next check indefinite integral: $D_x \sin(e^x) = [\cos(e^x)]\, D_x e^x = [\cos(e^x)]\, e^x$ $\boxed{\checkmark}$.
So $\int_0^{\ln(2\pi)} e^x \cos(e^x)\ dx = \sin e^x \,\Big|_{x=0}^{x=\ln(2\pi)} = \sin e^{\ln(2\pi)} - \sin e^0 = \sin(2\pi) - \sin 1 = 0 - \sin 1 = -\sin 1$

4. Evaluate

$$\int_{x=0}^{x=\frac{3\pi}{2}} e^x \cos x\ dx \ .$$

4soln. Below we show that

$$\int e^x \cos x\ dx = \frac{e^x(\sin x + \cos x)}{2} + C.$$

So

$$\int_{x=0}^{x=\frac{3\pi}{2}} e^{x}\,\cos x\;dx = \frac{e^{x}(\sin x+\cos x)}{2}\,\Big|_{0}^{3\pi/2} = \frac{e^{3\pi/2}(-1)}{2} - \frac{e^{0}(1)}{2} = \frac{-1-e^{3\pi/2}}{2}.$$

To find the indefinite integral, use two integration by parts and the *bring to the other side* idea. For the <u>two</u> integration by parts, put the expontential function with either the u's <u>both</u> times or the dv's <u>both</u> times.

$\boxed{\text{Way \# 1}}$

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{1x} \qquad\qquad dv_1 = \cos 1x\,dx$$

$$du_1 = 1e^{1x}\,dx \qquad\qquad v_1 = \frac{1}{1}\sin 1x\ .$$

So by integration by parts

$$\int e^{1x}\,\cos 1x\;dx \;=\; \frac{1}{1}e^{1x}\sin 1x - \frac{1}{1}\int e^{1x}\sin 1x\,dx\ .$$

Now let

$$u_2 = e^{1x} \qquad\qquad dv_2 = \sin 1x\,dx$$

$$du_2 = 1e^{1x}\,dx \qquad\qquad v_2 = \frac{-1}{1}\cos 1x\ .$$

to get

$$\begin{aligned}\int e^{1x}\,\cos 1x\;dx \;&=\; \frac{1}{1}e^{1x}\sin 1x - \frac{1}{1}\left[\frac{-1}{1}e^{1x}\cos 1x - \frac{-1}{1}\int e^{1x}\cos 1x\,dx\right]\\ &=\; \frac{1}{1}e^{1x}\sin 1x + \frac{1}{1^2}e^{1x}\cos 1x - \frac{1^2}{1^2}\int e^{1x}\cos 1x\,dx\ .\end{aligned}$$

Now solving for $\int e^{1x}\cos 1x\,dx$ (use the *bring to the other side* idea) we get

$$\left[1+\frac{1^2}{1^2}\right]\int e^{1x}\,\cos 1x\;dx \;=\; \frac{1}{1}e^{1x}\sin 1x + \frac{1}{1^2}e^{1x}\cos 1x + K$$

and so

$$\begin{aligned}\int e^{1x}\,\cos 1x\;dx \;&=\; \left[\frac{1^2}{2}\right]\left(\frac{1}{1}e^{1x}\sin 1x + \frac{1}{1^2}e^{1x}\cos 1x + K\right)\\ &=\; \frac{1}{2}\,e^{1x}\sin 1x + \frac{1}{2}\,e^{1x}\cos 1x + \left[\frac{K1^2}{2}\right]\\ &=\; \frac{e^{1x}}{2}\,(1\sin 1x + 1\cos 1x) + \left[\frac{K1^2}{2}\right]\ .\end{aligned}$$

Thus

$$\int e^{1x}\,\cos 1x\;dx \;=\; \boxed{\frac{e^{1x}}{2}\,(1\cos 1x + 1\sin 1x) + C}\,.$$

$\boxed{\text{Way \# 2}}$

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$u_1 = \cos 1x \qquad dv_1 = e^{1x} dx$$
$$du_1 = -1 \sin 1x \, dx \qquad v_1 = \frac{1}{1} e^{1x} \, .$$

So, by integration by parts

$$\int e^{1x} \cos 1x \; dx \;=\; \frac{1}{1} e^{1x} \cos 1x - \frac{-1}{1} \int e^{1x} \sin 1x \, dx \; .$$

Now let

$$u_2 = \sin 1x \qquad dv_2 = e^{1x} dx$$
$$du_2 = 1 \cos 1x \, dx \qquad v_2 = \frac{1}{1} e^{1x} \, .$$

to get

$$\begin{aligned} \int e^{1x} \cos 1x \; dx \;&=\; \frac{1}{1} e^{1x} \cos 1x + \frac{1}{1} \left[\frac{1}{1} e^{1x} \sin 1x - \frac{1}{1} \int e^{1x} \cos 1x \, dx \right] \\ &=\; \frac{1}{1} e^{1x} \cos 1x + \frac{1}{1^2} e^{1x} \sin 1x - \frac{1^2}{1^2} \int e^{1x} \cos 1x \, dx \; . \end{aligned}$$

Now solving for $\int e^{1x} \cos 1x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{1^2}{1^2} \right] \int e^{1x} \cos 1x \; dx \;=\; \frac{1}{1} e^{1x} \cos 1x + \frac{1}{1^2} e^{1x} \sin 1x + K$$

and so

$$\begin{aligned} \int e^{1x} \cos 1x \; dx \;&=\; \left[\frac{1^2}{1^2 + 1^2} \right] \left(\frac{1}{1} e^{1x} \cos 1x + \frac{1}{1^2} e^{1x} \sin 1x + K \right) \\ &=\; \frac{1}{2} \, e^{1x} \cos 1x + \frac{1}{2} \, e^{1x} \sin 1x + \left[\frac{K 1^2}{1^2 + 1^2} \right] \\ &=\; \frac{e^{1x}}{2} \, (1 \cos 1x + 1 \sin 1x) + \left[\frac{K 1^2}{1^2 + 1^2} \right] \end{aligned}$$

Thus

$$\int e^{1x} \cos 1x \; dx \;=\; \boxed{\frac{e^{1x}}{2} \, (1 \cos 1x + 1 \sin 1x) + C} \, .$$

$\boxed{\text{Doesn't Work Way}}$

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

5. Investigate the convergence of

$$\int_{x=1}^{x=\infty} \frac{1 - e^{-x}}{x} \; dx \; .$$

TABLE 8.5

b	$\int_1^b \frac{1-e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

EXAMPLE 9 Investigate the convergence of $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$.

Solution The integrand suggests a comparison of $f(x) = (1-e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty}(1-e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because $\displaystyle\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$.

5soln. This is Example 9 from § 8.8 of our book by Thomas.

6. Evaluate

$$\int_{x=0}^{x=1} \sin^4 x\,dx\ .$$

6soln. From Class Handout on Trig. Substitution: **Example 4.** $\int \sin^4 x\,dx$.

u-du sub does not work (why? e.g.: $\int \sin^4 x\,dx \overset{u=\cos x}{=} -\int \boxed{\sin^3 x}\,[-\sin x\,dx]$).

For $\int \sin^n x \cos^m x\,dx$, with BOTH $m, n \in \{0, 2, 4, 6, \ldots\}$, use the half-angle formulas.

$$\begin{aligned}
\int \sin^4 x\,dx &\overset{\text{alg.}}{=} \int [\sin^2 x]^2\,dx \overset{(\frac{1}{2}\angle)}{=} \int \left[\frac{1-\cos(2x)}{2}\right]^2 dx \overset{\text{alg.}}{=} \frac{1}{4}\int [1 - 2\cos(2x) + \cos^2(2x)]\,dx \\
&\overset{(\frac{1}{2}\angle)}{=} \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] dx \\
&= \frac{1}{4}\int dx - \frac{1}{4}\int 2\cos(2x)\,dx + \frac{1}{4}\cdot\frac{1}{2}\int dx + \frac{1}{4}\cdot\frac{1}{2}\int \cos(4x)\,dx \\
&= \frac{1}{4}\left(1+\frac{1}{2}\right)\int dx - \frac{1}{4}\cdot\int \cos(2x)\,[2dx] + \left(\frac{1}{4}\cdot\frac{1}{2}\right)\cdot\left(\frac{1}{4}\right)\int \cos(4x)\,[4dx] \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.
\end{aligned}$$

So, since $\sin 0 = 0$,

$$\begin{aligned}
\int_{x=0}^{x=1} \sin^4 x\,dx &= \left[\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right]\Big|_{x=0}^{x=1} \\
&= \left[\frac{3}{8} - \frac{1}{4}\sin 2 + \frac{1}{32}\sin 4\right] - [0 - 0 + 0] \\
&= \frac{3}{8} - \frac{1}{4}\sin 2 + \frac{1}{32}\sin 4.
\end{aligned}$$

7. Evaluate

$$\int_{x=5}^{x=10} \frac{\sqrt{x^2-25}}{x}\,dx$$

AND specify the initial substitution.

7soln. The integrand has a $u^2 - a^2$, so we let $u = a\sec\theta$. $\boxed{x = 5sec\theta \quad \text{so} \quad dx = 5\sec\theta\tan\theta\, d\theta}$

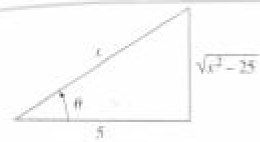
Thus,

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= 5\int \tan^2\theta\,d\theta \qquad \tan\theta \geq 0 \text{ since } 0 \leq \theta < \pi/2$$

$$= 5\int (\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C$$



$x = 5\sec\theta$

Figure 8.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 8.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

• Check $D_x\left[(x^2-25)^{1/2} - 5\sec^{-1}\left(\frac{x}{5}\right)\right]$

$$= \frac{1}{2}(x^2-25)^{-1/2}(2x) - 5\,\frac{1}{\left|\frac{x}{5}\right|\sqrt{\left(\frac{x}{5}\right)^2 - 1}}\cdot\frac{1}{5}$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{1}{\frac{x}{5}\sqrt{\frac{x^2}{25} - \frac{25}{25}}} \qquad (\text{know } x \geq 5)$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{25}{x(x^2-25)^{1/2}} = \frac{(x^2-25)}{x(x^2-25)^{1/2}}\cdot\frac{(x^2-25)^{1/2}}{(x^2-25)^{1/2}}$$

$$= \frac{(x^2-25)(x^2-25)^{1/2}}{x(x^2-25)} = \frac{\sqrt{x^2-25}}{x}. \checkmark$$

• $$\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)\Big|_{x=5}^{x=10}$$

$$= \left[\sqrt{100-25} - 5\sec^{-1}2\right] - \left[0 - \cancelto{0}{5\sec^{-1}1}\right] =$$

$$= \sqrt{75} - 5\cdot\frac{\pi}{3} = 5\sqrt{3} - 5\left(\frac{\pi}{3}\right) = 5\left(\sqrt{3} - \frac{\pi}{3}\right).$$

8. Evaluate

$$\int_{x=1}^{x=3} \frac{5x^2+3x-2}{x^3+2x^2}\,dx\ .$$

8soln.

- $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2)+B(x+2)+Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5=A+C \Rightarrow A=2$. So $\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x+2}\right)dx = 2\ln|x|+\frac{1}{x}+3\ln|x+2|+C$.

- Check $D_x\left[2\ln|x| + x^{-1} + 3\ln|x+2|\right] = \frac{2}{x} + -1x^{-2} + \frac{3}{x+2}$

$$= \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2} = \frac{2x(x+2)-(x+2)+3x^2}{x^2(x+2)} = \frac{5x^2+3x-2}{x^3+2x^2}\ ✓$$

- $\left[3\ln|x+2| + 2\ln x + \frac{1}{x}\right]\Big|_{x=1}^{x=3} =$

$$\left[3\ln 5 + 2\ln 3 + \tfrac{1}{3}\right] - \left[3\ln 3 + \underbrace{2\ln 1}_{=0} + 1\right] =$$

$$3\ln 5 - \ln 3 - \tfrac{2}{3}\ .$$

9. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}\,.$$

Answer:

9soln.

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \tan^{-1} x \Big]_a^0 \\
&= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\
&= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\
&= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15). ■

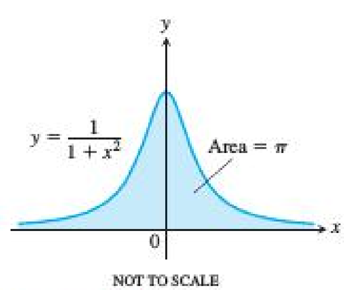


FIGURE 8.15 The area under this curve is finite (Example 2).

10. For which value of p does

$$\int_0^1 \frac{1}{x^p}\,dx = 1.25\ ?$$

10soln. First compute

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0+}\int_t^1 x^{-p}\,dx \overset{p\neq 1}{=} \lim_{t\to 0+}\frac{x^{1-p}}{1-p}\Big|_t^1 = \frac{1}{1-p}\lim_{t\to 0+} x^{1-p}\Big|_t^1 = \frac{1}{1-p}\lim_{t\to 0+}\left[1-t^{1-p}\right].$$

If $1-p>0$, or equivalently $1>p$, then $\lim_{t\to 0+} t^{1-p}=0$ and so

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{t\to 0+}\left[1-t^{1-p}\right] = \frac{1}{1-p}[1-0] = \frac{1}{1-p}.$$

If $1-p<0$, or equivalently $1<p$, then $\lim_{t\to 0+} t^{1-p}=\infty$ and so

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{t\to 0+}\left[1-t^{1-p}\right] = \infty.$$

If $p=1$, then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0+}\ln x\Big|_t^1 = \lim_{t\to 0+}[0-\ln t] = \infty.$$

So we need

$$p<1 \quad\text{and}\quad \frac{1}{1-p}=1.25.$$

Note

$$\frac{1}{1-p}=1.25=\frac{5}{4}=\frac{1}{\frac{4}{5}} \quad\Leftrightarrow\quad 1-p=\frac{4}{5} \quad\Leftrightarrow\quad p=1-\frac{4}{5}=\frac{1}{5}=0.2.$$

11. Evaluate the integral

$$\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx\ .$$

11soln.

• $\int x^{-3}\,dx = \frac{x^{-2}}{-2}+C$

$$\int_{x=0}^{x=1} x^{-3}\,dx = \lim_{a\to 0^+}\frac{x^{-2}}{-2}\Big|_{x=a}^{x=1} = \frac{1}{2}\lim_{a\to 0^+}\left[\frac{1}{x^2}\right]_{x=1}^{x=a} =$$

$$\frac{1}{2}\lim_{x\to 0^+}\left[\frac{1}{a^2}-1\right] = \infty.$$ Similarly, $\int_{-1}^{0} x^{-3}\,dx = -\infty$

• $\int_{-1}^{1} x^{-3}\,dx = \int_{-1}^{0} x^{-3}\,dx + \int_0^1 x^{-3}\,dx = -\infty+\infty$ so DNE.

12. Limit of a sequence. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{25n^3+4n^2+n-5}}{7n^{\frac{3}{2}}+6n-1} \,.$$

12soln.

$$\frac{\sqrt{25n^3+4n^2+n-5}}{7n^{3/2}+6n-1} = \frac{\sqrt{25+\frac{4}{n}+\frac{1}{n^2}-\frac{5}{n^3}}}{7+\frac{6}{n^{1/2}}-\frac{1}{n^{3/2}}} \xrightarrow{n\to\infty} \frac{\sqrt{25}}{7} = \frac{5}{7}.$$

$\boxed{\div \text{ num \& dem. by } n^{3/2} = \sqrt{n^3}}$

13. Consider the formal series

$$\sum_{n=1}^{\infty} \frac{1}{n+3^n}.$$

13soln. This is the first problem from the 38 Serious Series Problems.

$\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ for all $n \geq 1$. $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent geometric series $\left[|r| = \frac{1}{3} < 1\right]$, so $\sum_{n=1}^{\infty}\frac{1}{n+3^n}$ converges by the Comparison Test.

☹Note that $\frac{1}{n+3^n} \neq \frac{1}{n}+\frac{1}{3^n}$ ... unfortunately, some folks told me they were equal. ☹

14. Consider the formal series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+2)(n+7)} \,.$$

14soln. $\frac{1}{(n+2)(n+7)} \overset{n \text{ big}}{\sim} \frac{1}{(n)(n)} = \frac{1}{n^2}$. So let $b_n = \frac{1}{n^2}$ and $a_n = \frac{(-1)^n}{(n+2)(n+7)}$. Then

$$\lim_{n\to\infty}\frac{|a_n|}{b_n} = \lim_{n\to\infty}\frac{1}{(n+2)(n+7)}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{n^2}{(n+2)(n+7)}$$

$$= \lim_{n\to\infty}\frac{\frac{n^2}{n^2}}{\left(\frac{n+2}{n}\right)\left(\frac{n+7}{n}\right)} \lim_{n\to\infty}\frac{1}{\left(1+\frac{2}{n}\right)\left(1+\frac{7}{n}\right)} = \frac{1}{(1+0)(1+0)} = 1 \,.$$

Since $0 < \lim_{n\to\infty}\frac{|a_n|}{b_n} < \infty$, by the LCT, $\sum b_n$ and $\sum |a_n|$ *do the same thing.*
We know $\sum b_n = \sum \frac{1}{n^2}$ (p-series, $p = 2 > 1$ so) converges.
So $\sum |a_n|$ converges. So $\sum a_n$ is absolutely convergent.

15. By using the Limit Comparison Test, one can show that the formal series

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}} \,. \tag{15}$$

is:

15soln. Let

$$a_n = \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}}$$

For n sufficiently big,

$$a_n \;=\; \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}} \;\overset{\text{when } n \text{ is big}}{\approx}\; \frac{n}{\sqrt{(n)\,(n)\,(n)\,(n)}} \;=\; \frac{n^1}{n^{4/2}} \;=\; \frac{1}{n}\;.$$

So we let $b_n = \frac{1}{n}$ and compute

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}}\,\frac{n}{1} = \frac{n^2}{[\,(n+1)\,(n+2)\,(n+3)\,(n+4)]^{1/2}} \\ &= \left[\frac{n^4}{(n+1)\,(n+2)\,(n+3)\,(n+4)}\right]^{1/2} = \left[\frac{n}{(n+1)}\,\frac{n}{(n+2)}\,\frac{n}{(n+3)}\,\frac{n}{(n+4)}\right]^{1/2} \\ &\xrightarrow{n\to\infty} [\,(1)\;(1)\;(1)\;(1)\,]^{1/2} \;=\; 1\;.\end{aligned}$$

Since $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, the LCT says the $\sum a_n$ and $\sum b_n$ do the same thing. Since $\sum b_n$ is a p-series with $p = 1 \le 1$, the $\sum b_n$ diverges. So the $\sum a_n$ diverges.

16. Find **all** real numbers r satisfying that

$$\sum_{n=2}^{\infty} r^n \;=\; \frac{1}{6}\;.$$

16soln. Soln: $\dfrac{-1}{2}$ and $\dfrac{1}{3}$

First note that for the series $\sum_{n=2}^{\infty} r^n$ to converge (so that the problem even makes sense), we need

$$|r| < 1.$$

So let $\boxed{|r| < 1}$. Next, to find the sum $\sum_{n=2}^{\infty} r^n$, consider the partial sums $s_n \overset{\text{def}}{=} r^2 + r^3 + \ldots + r^{n-1} + r^n$.

Cancellation Heaven occurs with a geometric series when one computes $s_n - rs_n$. Let's see why.

$$\begin{aligned} s_n &= r^2 + r^3 + \ldots + r^{n-1} + r^n \\ r\,s_n &= r^3 + r^4 + \ldots + r^n + r^{n+1}\end{aligned}$$

Do you see the cancellation that would occur if we take $s_n - rs_n$?

$$\begin{aligned} s_n &= r^2 + \cancel{r^3} + \ldots + \cancel{r^{n-1}} + \cancel{r^n} \\ r\,s_n &= \cancel{r^3} + \cancel{r^4} + \ldots + \cancel{r^n} + r^{n+1}\end{aligned}$$

substract

$$(1-r)\;s_n \;\overset{\text{Ⓐ}}{=}\; s_n - rs_n \;=\; r^2 \qquad - r^{n+1}$$

and since $r \neq 1$, then

$$s_n \;=\; \frac{r^2 - r^{n+1}}{1-r} \;\xrightarrow{\text{since } |r|<1}\; \frac{r^2}{1-r} = \sum_{n=2}^{\infty} r^n.$$

So we are looking for $r \in \mathbb{R}$ so that $|r| < 1$ and $\frac{r^2}{1-r} = \frac{1}{6}$. Note $\left[\frac{r^2}{1-r} = \frac{1}{6}\right] \Leftrightarrow [6r^2 = 1 - r] \Leftrightarrow [6r^2 + r - 1 = 0] \Leftrightarrow$

$$r = \frac{-1 \pm \sqrt{1 + 4\,(6)}}{2\,(6)} = \frac{-1 \pm 5}{12} = \begin{cases} \frac{-1+5}{12} = \frac{4}{12} = \frac{1}{3} \\ \frac{-1-5}{12} = -\frac{6}{12} = -\frac{1}{2}. \end{cases}$$

17. The series

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n},$$

is

17soln.

converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x)+1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test, $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x}\right) dx$

$= \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \lim_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges

18. What is the smallest integer N such that the Alternating Series Estimate/Remainder Theorem guarentees that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \leq 0.05?$$

Note that $0.05 = \frac{0.05}{1.0000} = \frac{5}{100} = \frac{1}{20}$.

18soln. Note that $0 \leq \frac{1}{n^2} \searrow 0$ so the AST applies and tells us that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges and that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \leq \frac{1}{(N+1)^2}.$$

So

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \overset{\text{have}}{\leq} \frac{1}{(N+1)^2} \overset{\text{want}}{\leq} \frac{1}{20}.$$

Note

$$\left[\frac{1}{(N+1)^2} \leq \frac{1}{20} \right] \Leftrightarrow \left[20 \leq (N+1)^2 \right].$$

If $N = 3$, then $(N+1)^2 = (3+1)^2 = 4^2 = 16 < 20$.
If $N = 4$, the $(N+1)^2 = (4+1)^2 = 5^2 = 25 \geq 20$.

19. Find the 4th order Maclaurin polynonial for

$$f(x) = \frac{1}{(x+1)^3}.$$

19soln. ans: $P_4(x) = 1 - 3x + 6x^2 - 10x^3 + 15x^4$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \frac{f^{(n)}(0)}{n!}$
0	$(x+1)^{-3}$	1	$\frac{1}{0!} = 1$
1	$-3(x+1)^{-4}$	-3	$\frac{-3}{1!} = -3$
2	$+3\cdot4(x+1)^{-5}$	$+3\cdot4$	$\frac{3\cdot4}{2!} = \frac{3\cdot\cancel{4}^{2}}{\cancel{2}} = 6$
3	$-3\cdot4\cdot5\,(x+1)^{-6}$	$-3\cdot4\cdot5$	$\frac{-3\cdot4\cdot5}{3!} = \frac{-\cancel{3}\cdot\cancel{4}^{2}\cdot5}{1\cdot\cancel{2}\cdot\cancel{3}} = -10$
4	$+3\cdot4\cdot5\cdot6\,(x+1)^{-7}$	$+3\cdot4\cdot5\cdot6$	$\frac{3\cdot4\cdot5\cdot6}{4!} = \frac{\cancel{3}\cdot\cancel{4}\cdot5\cdot\cancel{6}^{3}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}} = 15$

Note: Use the term Maclaurin so we know the center is $x_0 = 0$.

20. Find the 10th-order Taylor polynomial $y = P_{10}(x)$, centered about $x_0 = 17$, of the function

$$f(x) = 5 + 6x^7.$$

20soln. ans: $P_{10}(x) = 5 + 6x^7$

Note $f^{(n)}(x) = 0$ for each $n \geq 8$ and $x \in \mathbb{R}$. Let's following notation from Probelm **0B**. If $N \geq 7$, then $f^{(N+1)}(c) = 0$ for any $c \in \mathbb{R}$ and so

$$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!}(x-17)^{N+1} \right| = \frac{0}{(N+1)!}|x-17|^{N+1} = 0\ ,$$

and so $P_N(x) = f(x)$. So $p_{10}(x) = f(x)$.

21. Find a power series representation for

$$f(x) = \ln(10 - x).$$

Hint: $\ln(ab) = \ln a + \ln b$.

21soln. ans: $\ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{n\,10^n}$

$$\ln(10-x) = \ln\left(10\left(1-\frac{x}{10}\right)\right) = \ln 10 + \ln\left[1+\left(\frac{-x}{10}\right)\right]$$ ← used a "Commonly used Taylor Series"

$$= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{-x}{10}\right)^n = \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n \frac{x^n}{10^n}$$

$$= \ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{n\,10^n}$$

$$(-1)^{n+1}(-1)^n = (-1)^{2n+1} = (-1)^{2n}(-1)^1 = -1.$$

I forgot to ask when is this expansion valid so let's do:

Valid $\iff -1 < -\frac{x}{10} \leq 1 \iff -10 \leq x < 10$

22. Consider the function $f(x) = e^x$ over the interval $(-1, 3)$. The 4th order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = \sum_{n=0}^{4} \frac{x^n}{n!}.$$

The 4th order Remainder term $R_4(x)$ is defined by $R_4(x) = f(x) - P_4(x)$ and so $e^x \approx P_4(x)$ where the approximation is within an error of $|R_4(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_4(x)|$ that is valid for each $x \in (-1, 3)$.

22soln. ans: $\dfrac{(e^3)(3^5)}{5!}$

For each $x \in (-1,3)$, the exists c between x & x_0 (so c is also in $(-1,3)$) so that:

$$|R_4(x)| \overset{\text{big}}{\underset{\text{theorm}}{=}} \left|\frac{f^{(5)}(c)}{5!}x^5\right| = \frac{e^c |x|^5}{5!} \leq \frac{e^3 \cdot 3^5}{5!}$$

23. Suppose that the interval of convergence of the series $\sum_{n=1}^{\infty} c_n (x - x_0)^n$ is $(x_0 - R, x_0 + R]$. What can be said about the series at $x = x_0 + R$?

23soln. ans: It must be conditionally convergent.

63. Prove: If the interval of convergence of the series $\sum_{k=0}^{\infty} c_k (x - x_0)^k$ is $(x_0 - R, x_0 + R]$, then the series converges conditionally at $x_0 + R$.

63. The assumption is that $\sum_{k=0}^{\infty} c_k R^k$ is convergent and $\sum_{k=0}^{\infty} c_k (-R)^k$ is divergent. Suppose that $\sum_{k=0}^{\infty} c_k R^k$ is absolutely convergent then $\sum_{k=0}^{\infty} c_k (-R)^k$ is also absolutely convergent and hence convergent because $|c_k R^k| = |c_k (-R)^k|$, which contradicts the assumption that $\sum_{k=0}^{\infty} c_k (-R)^k$ is divergent so $\sum_{k=0}^{\infty} c_k R^k$ must be conditionally convergent.

24. Suppose that the power series $\sum_{n=1}^{\infty} c_n (x - 10)^n$ has interval of convergence $(1, 19)$. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} c_n x^{2n}$?

24soln. ans: $(-3, 3)$

$\Sigma c_n x^{2n}$ converges $\Leftrightarrow \Sigma c_n (x^2)^n$ conv. $\Leftrightarrow \Sigma c_n [(x^2+10)-10]^n$ conv.

$\xLeftrightarrow{\text{given}}$ $+1 < x^2 + 10 < 19 \Leftrightarrow -9 < x^2 < 9 \Leftrightarrow -3 < x < 3.$

25. Describe the motion of a puffo whose position (x, y) is parameterized by

$$x = 6 \sin t$$
$$y = 3 \cos t$$

for $0 \le t \le 2\pi$.

25soln. ans: Moves once clockwise along the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ starting and ending at $(0, 3)$.

Since $\left[\frac{x(t)}{6}\right]^2 + \left[\frac{y(t)}{3}\right]^2 = [\sin t]^2 + [\cos t]^2 = 1$,
the puffo is moving along the ellipse $\left[\frac{x}{6}\right]^2 + \left[\frac{y}{3}\right]^2 = 1$, i.e., along the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$.
He starts at $(x(0), y(0)) = (6 \sin 0, 3 \cos 0) = (0, 3)$.
He finishes at $(x(2\pi), y(2\pi)) = (6 \sin 2\pi, 3 \cos 2\pi) = (0, 3)$.
As he moves from $t = 0$ to $t = 2\pi$, he traces out the ellipse one time.
To figure out if he is moving CW or CCW, note $\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = \left(6 \sin \frac{\pi}{2}, 3 \cos \frac{\pi}{2}\right) = (6, 0)$.
So Mr. Puffo is moving clockwise.

26. Eliminate the parameter to find a Cartesian equation of the curve

$$x = 5e^t$$
$$y = 21e^{-t}.$$

26soln. ans: $y = \dfrac{105}{x}$

$y(t) = 21e^{-t} = \frac{21}{e^t} = \frac{(5)(21)}{5e^t} = \frac{105}{x(t)}$.

27. Find the arc length of the curve

$$x = 3t^2$$
$$y = 2t^3$$

for $0 \le t \le 3$

27soln. ans: $20\sqrt{10} - 2$

$$\begin{aligned}
\text{AL} &= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t=0}^{t=3} \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_{t=0}^{t=3} \sqrt{6^2t^2 + 6^2t^4}\, dt \\
&= \int_{t=0}^{t=3} \sqrt{6^2t^2\left(1+t^2\right)}\, dt = \int_{t=0}^{t=3} 6t\sqrt{1+t^2}\, dt = 3\int_{t=0}^{t=3} (2t)\left(1+t^2\right)^{1/2}\, dt \\
&\underset{\text{so } du=2t\,dt}{\overset{\text{let } u=1+t^2}{=}} 3\int_{u=1}^{u=10} u^{1/2}\, du = (3)\left(\frac{2}{3}\right) u^{3/2}\Big|_{u=1}^{u=10} = 2u\sqrt{u}\,\Big|_{u=1}^{u=10} = 20\sqrt{10} - 2.
\end{aligned}$$

28. Find a parameterization for the line segment from $(-1, 2)$ to $(10, -6)$ for $0 \le t \le 1$.

28soln. ans: $x = -1 + 11t$ and $y = 2 - 8t$

$$\begin{aligned}
x(t) &= -1 + \left(10 - \left({}^{-}1\right)\right) t = -1 + 11t \\
y(t) &= 2 + \left({}^{-}6 - 2\right) t \qquad = 2 - 8t.
\end{aligned}$$

29. Find an equation of the tangent line to the curve at the point corresponding to $t = 11\pi$.

$$x = t\sin t$$
$$y = t\cos t.$$

29soln. ans: $y = \frac{x}{11\pi} - 11\pi$

$(x(11\pi), y(11\pi)) = (0, -11\pi)$

$$\left.\frac{dy}{dx}\right|_{t=11\pi} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=11\pi} = \left.\frac{\cos t - t\sin t}{\sin t + t\cos t}\right|_{t=11\pi} = \frac{-1-0}{0-11\pi} = \frac{1}{11\pi}.$$

So equation of tangent line to curve when $t = 11\pi$ is

$$\begin{aligned}
\left(y - {}^{-}11\pi\right) &= \frac{1}{11\pi}(x - 0) \\
y + 11\pi &= \frac{1}{11\pi}x \\
y &= \frac{1}{11\pi}x - 11\pi.
\end{aligned}$$

30. Prof. Girardi likes

$\boxed{\text{Good Luck in your math fun to come!}}$