MARK BOX		
PROBLEM	POINTS	
0	10	
1	10	
2	5	
3	10	
4–16	65=13x5	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  1. HAND IN PART. Hand in only this part.
  2. STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
     You can take this part home to learn from and to check your answers once the solutions are posted.
- On Problem 0, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold during the exam (it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets.
- Cheating is grounds for a F in the course.
- At a student's request, I will project my watch upon the projector screen.
  Upon request, you will be given as much (blank) scratch paper as you need.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.7–10.10, 11.1, 11.2 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

0. Fill-in the boxes.

0A. Power Series. Consider a (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \, (x - x_0)^n \;, \tag{0A}$$

with radius of convergence $R \in [0, \infty]$. (Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)
Without any other further information of $\{a_n\}_{n=0}^{\infty}$, answer the following questions.
The choices for the next 4 boxes are: **AC**, **CC**, **DIVG**, or **anything**.

AC stands for: *is always absolutely convergent.*
CC stands for: *is always conditionally convergent.*
DIVG stands for: *is always divergent.*
anything stands for: *can do anything* i.e., there are examples showing that it can AC, CC, or DIVG.

0A.1. At the center $x = x_0$, the power series in (0A) [AC].

0A.2. For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series in (0A) [AC].

0A.3. For $x \in \mathbb{R}$ such that $|x - x_0| > R$, the power series in (0A) [DIVG].

0A.4. If $0 < R < \infty$, then for the endpoints $x = x_0 \pm R$, the power series in (0A) [anything].

0A.5. Furthermore, if α and β are in the interval $(x_0 - R, x_0 + R)$, then (Hint: note the $\Big|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta}$ already in there.)

$$\int_{x=\alpha}^{x=\beta} h(x)\, dx \;=\; \sum_{n=0}^{\infty} \boxed{\frac{a_n}{n+1}\;(x - x_0)^{n+1}} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

0B. Taylor/Maclaurin Polynomials and Series.
Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

0B.1. The n^{th} Taylor coefficient of $y = f(x)$ about x_0 is

$$c_n \;=\; \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

0B.2. The N^{th}-order Taylor polynomial of $y = f(x)$ about x_0, in open form (so with $\ldots$ and without a $\sum$-sign), is

$$P_N(x) \;=\; \boxed{f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N}$$

0B.3. The Taylor series of $y = f(x)$ about x_0, in closed form (so, with a $\sum$-sign and without $\ldots$), is

$$P_\infty(x) \;=\; \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\;(x - x_0)^n}$$

0C. Parametric Curves. Consider the curve $\mathcal{C}$ parameterized by

$$x = x(t)$$
$$y = y(t)$$

for $a \le t \le b$.

0C.1. Express $\frac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\frac{dy}{dx} =$ $\boxed{\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}}$

0C.2. The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

Arc Length = $\boxed{\displaystyle\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$

1. Commonly Used Taylor Series

Fill in Problem 1's blank boxes with the choices a – ℓ, which are provided below.
You may use a choice more than once or not at all.
A sample question is already done for you.

a. $\displaystyle\sum_{n=0}^{\infty} x^n$ **d.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ **g.** $x \in \mathbb{R}$ **j.** $(-1, 1]$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ **e.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ **h.** $(-1, 1)$ **k.** $[-1, 1)$

c. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ **f.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ **i.** $[-1, 1]$ **ℓ.** none of the others

sample. A power series expansion for $y = \frac{1}{1-x}$ is [a] and is valid precisely when [h].

1.1. A power series expansion for $y = \cos x$ is [d] and is valid precisely when [g].

1.2. A power series expansion for $y = \sin x$ is [e] and is valid precisely when [g].

1.3. A power series expansion for $y = e^x$ is [b] and is valid precisely when [g].

1.4. A power series expansion for $y = \ln(1 + x)$ is [c] and is valid precisely when [j].

1.5. A power series expansion for $y = \tan^{-1} x$ is [f] and is valid precisely when [i].

2. Identify a parametrization of a circle with center at $(0,0)$ and radius 1, which is traced in the various below described fashions, by fill in the blanks boxes with the below choices A – I.
You may use a choice more than once or not at all.

2.1. traced once, starting at $(1,0)$, in the counterclockwise direction. A parameterization is [A]

2.2. traced once, starting at $(1,0)$, in the clockwise direction. A parameterization is [B]

2.3. traced once, starting at $(-1,0)$, in the counterclockwise direction. A parameterization is [D]

2.4. traced once, starting at $(-1,0)$, in the clockwise direction. A parameterization is [C]

2.5. traced once, starting at $(0,-1)$, in the counterclockwise direction. A parameterization is [F]

$$\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{A}$$

$$\begin{aligned} x(t) &= \cos t \\ y(t) &= -\sin t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{B}$$

$$\begin{aligned} x(t) &= -\cos t \\ y(t) &= \sin t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{C}$$

$$\begin{aligned} x(t) &= -\cos t \\ y(t) &= -\sin t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{D}$$

$$\begin{aligned} x(t) &= \sin t \\ y(t) &= \cos t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{E}$$

$$\begin{aligned} x(t) &= \sin t \\ y(t) &= -\cos t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{F}$$

$$\begin{aligned} x(t) &= -\sin t \\ y(t) &= \cos t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{G}$$

$$\begin{aligned} x(t) &= -\sin t \\ y(t) &= -\cos t \end{aligned} \qquad \text{for } 0 \le t \le 2\pi \tag{H}$$

None of the other choices. (I)

2soln. Note that for each of the given choices of a parameterization, $[x(t)]^2 + [y(t)]^2 = 1$ and so the puffo is indeed running along, in some fashion, the curve $x^2 + y^2 = 1$, which is a circle with center $(0,0)$ and radius 1. Plugging in $t = 0$ will give you the starting point. Then use the behaviour of the trig functions involved to get the direction.

3. Taylor's Remainder Theorem.

3a. Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Thus, $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Remainder Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{(N+1)}} \quad \text{for some } c \text{ between } \boxed{x} \text{ and } \boxed{x_0} .$$

3b. **In this problem, you must show your work and clearly explain your thought process.**
Using Taylor's (Big) Remainder Theorem (and not using the facts on the *Commonly Used Taylor Series* handout), show that

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2 .$$

▷. Hint. Indeed, the facts listed on the *Commonly Used Taylor Series* are shown by using Taylor's Remainder Theorem so you can think of this problem as verifying/showing one of these facts listed on the *Commonly Used Taylor Series* handout.

▷. Hint. Consider the function $f(x) = e^x$. Note that then $f(2) = e^2$.

3bsoln.

Let $f(x) = e^x$ with $f\colon \mathbb{R} \to \mathbb{R}$. Take the center $x_0 = 0$. Let's follow the notation from problem **0B**. So $f^{(n)}(x) = e^x$ and $f^{(n)}(x_0) = e^0 = 1$ for each $n \in \mathbb{N} \cup \{0\}$. So

$$e^x = P_N(x) + R_N(x)$$

and

$$e^x = P_\infty(x) \quad \text{if and only if} \quad \lim_{N\to\infty} |R_N(x)| = 0$$

where

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=0}^{N} \frac{x^n}{n!}$$

and $P_\infty(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and Taylor's Remainder Theorem tells us that

$$\text{for some } c \text{ between } x \text{ and } x_0\text{:} \quad R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{(N+1)} = \frac{e^c x^{N+1}}{(N+1)!} . \tag{13.7}$$

Taking $x = 2$ in (13.7) gives that for some c between 2 and 0:

$$|R_N(2)| = \frac{e^c\, |x|^{N+1}}{(N+1)!} = \frac{e^c 2^{N+1}}{(N+1)!} \overset{0\le c\le 2}{\le} \frac{e^2\, 2^{N+1}}{(N+1)!} \tag{13.8}$$

Next we want to show that the (good) upper bound we found in (13.8) tends to zero as $N \to \infty$. So we want to show that $\lim_{N\to\infty} \frac{e^2 2^{N+1}}{(N+1)!} = 0$. (Sometimes this step is easy but in this example we will have to use a little trick (tool) that sometimes works … here we go). Let $a_N = \frac{e^2 2^{N+1}}{(N+1)!}$. (To show that $\lim_N a_N = 0$, we will actually show something stronger, namely $\sum a_N$ converges.) The Ratio Test tells us that the series $\sum_{n=0}^{\infty} \frac{e^2\, 2^{n+1}}{(n+1)!}$ is (absolutely) convergent since applying the Ratio Test we get

$$\rho = \lim_{N\to\infty} \left| \frac{a_{N+1}}{a_N} \right| = \lim_{N\to\infty} \frac{e^2\, 2^{N+2}}{(N+2)!} \cdot \frac{(N+1)!}{e^2\, 2^{N+1}} = \lim_{N\to\infty} \frac{2}{N+2} = 0.$$

The n^{th} term test for divergence gives that if the series $\sum_n a_n$ converges, then the limit of the sequence $\{a_n\}_n$ is 0, i.e. $\lim_{n\to\infty} a_n = 0$. So

$$\lim_{N\to\infty} \frac{e^2\, 2^{N+1}}{(N+1)!} = 0 . \tag{13.9}$$

So

$$0 \le |R_N(2)| \overset{\text{by (13.8)}}{\le} \frac{e^2\, 2^{N+1}}{(N+1)!} \xrightarrow{\text{as } N\to\infty, \text{ by (13.9)}} 0.$$

The Squeeze/Sandwich Theorem gives that $\lim_{N\to\infty} |R_N(2)| = 0$.
So $e^2 = P_\infty(2)$, i.e.,

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} .$$

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
14	14a	14b	14c	14d	14e					
15	15a	15b	15c	15d	15e					
16	16a	16b	16c	16d	16e					
							5	2	0	0
							Total:			

STATEMENT OF MULTIPLE CHOICE PROBLEMS

These sheets of paper are <u>not</u> collected.

4. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} c_n x^n$, which is valid in some interval $(-R, R)$ where $R > 0$.

4soln. If a function can be represented by a power series centered at 0 on some interval $(-R, R)$, with $R > 0$, then that power series must be the Taylor series centered at 0. So $c_0 = \frac{f^{(0)}(0)}{0!} = f(0)$.

5. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} a_n x^n$, which is valid in some interval J containing 0 and the raduis of J striclty positive. Consider the two statements:
(A) If $y = f(x)$ is an even function (i.e., $f(-x) = f(x)$), then $a_1 = a_3 = a_5 = \cdots = 0$.
(B) If $y = f(x)$ is an odd function (i.e., $f(-x) = -f(x)$), then $a_0 = a_2 = a_4 = \cdots = 0$.

5soln. Both (A) and (B) are true.

*2.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-R, R)$.

a. Show that if f is even, then $a_1 = a_3 = a_5 = \cdots = 0$, i.e., the Taylor series for f at x = 0 contains only even powers of x.

b. Show that if f is odd, then $a_0 = a_2 = a_4 = \cdots = 0$, i.e., the Taylor series for f at x = 0 contains only odd powers of x.

It is known that all power series that converge to a function f(x) on an interval $(-R,R)$ are the same. This is a key property of power series that will be needed to complete this proof.

a. If f(x) is even, then $f(-x) =$ (1) ________.

Substitute $-x$ for x in the series $\sum_{n=0}^{\infty} a_n x^n$. What are the coefficients of the resulting power series for odd n?

The coefficients for odd n are (2) ________.

How does this show that the Taylor series for an even function f at x = 0 contains only even powers of x?

- (●) **A. The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $-a_n$. The only solution to $a_n = -a_n$ is $a_n = 0$.**
- ( ) **B.** The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $2a_n$. The only solution to $a_n = 2a_n$ is $a_n = 0$.
- ( ) **C.** The substitution of $-x$ resulted in a coefficient of 0 for all odd n, so the statement has been proven.
- ( ) **D.** The coefficients of the odd-n terms in the series for $f(-x)$ must equal both a_n and $\frac{1}{2}a_n$. The only solution to $a_n = \frac{1}{2}a_n$ is $a_n = 0$.

b. If f(x) is odd, then $f(-x) =$ (3) ________.

Substitute $-x$ for x in the series $\sum_{n=0}^{\infty} a_n x^n$. What are the coefficients of the resulting power series for even n?

The coefficients for even n are (4) ________.

How does this show that the Taylor series for an odd function f at x = 0 contains only odd powers of x?

- ( ) **A.** The substitution of $-x$ resulted in a coefficient of 0 for all even n, so the statement has been proven.
- ( ) **B.** The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $\frac{1}{2}a_n$. The only solution to $a_n = \frac{1}{2}a_n$ is $a_n = 0$.
- ( ) **C.** The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $2a_n$. The only solution to $a_n = 2a_n$ is $a_n = 0$.
- (●) **D. The coefficients of the even-n terms in the series for $f(-x)$ must equal both a_n and $-a_n$. The only solution to $a_n = -a_n$ is $a_n = 0$.**

(1) (●) **f(x)** ( ) −f(x)

(2) ( ) a_n (●) $\mathbf{-a_n}$ ( ) 0 ( ) $2a_n$ ( ) $\frac{1}{2}a_n$

(3) ( ) f(x) (●) **−f(x)**

(4) (●) $\mathbf{a_n}$ ( ) $-a_n$ ( ) 0 ( ) n ( ) $\frac{1}{2}a_n$

ID: 9.9.52

►. **Problems 4 and 5 were meant to help you with Problem 1. ☺☺☺**

6. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{10^n}$$

Recall that the interval of convergence is the set of x's for which the power series converges, either absolutely or conditionally.

6soln. The interval of convergence is $(-8, 12)$.

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(x-2)^n}\right|<1 \Rightarrow \frac{|x-2|}{10}<1 \Rightarrow |x-2|<10 \Rightarrow -10<x-2<10 \Rightarrow -8<x<12$; when $x=-8$ we have $\sum_{n=1}^{\infty}(-1)^n$, a divergent series; when $x=12$ we have $\sum_{n=1}^{\infty}1$, a divergent series

(a) the radius is 10; the interval of convergence is $-8<x<12$
(b) the interval of absolute convergence is $-8<x<12$
(c) there are no values for which the series converges conditionally

7. What is the LARGEST interval for which the power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

is absolutely convergent?

7soln.

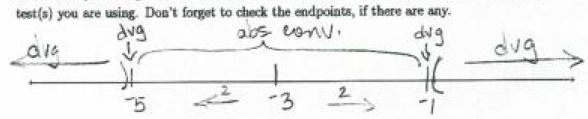
Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

Hint: $(2x+6)^n = [2(x+3)]^n = 2^n(x+3)^n = 2^n(x-(-3))^n$

The center is $x_0 =$ <u>-3</u> and the radius of convergence is $R =$ <u>2</u>.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



Ratio Test: $\rho = \lim_{n\to\infty}\left|\frac{(2x+6)^{n+1}}{4^{n+1}}\ \frac{4^n}{(2x+6)^n}\right| = \lim_{n\to\infty}\frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

or

Root Test: $\rho = \lim_{n\to\infty}\left|\frac{(2x+6)^n}{4^n}\right|^{1/n} = \lim_{n\to\infty}\frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$\rho<1 \iff |2x+6|<4 \iff 2|x+3|<4 \iff |x+3|<2 \iff |x-(-3)|<2$

endpts: $-3+2=-1$ and $-3-2=-5$

Check endpts

$x=-1$: $\sum_{n=1}^{\infty}\frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty}\frac{4^n}{4^n} = \sum_{n=1}^{\infty}1 = 1+1+1+\dots = \infty$ dvg

$x=-5$: $\sum_{n=1}^{\infty}\frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty}\frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty}\frac{(-1\cdot 4)^n}{4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n4^n}{4^n}$

$= \sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+1-1+\dots$

osc btw -1 & 0 $\Rightarrow$ dvg

8. Suppose that the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is 16. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{2n}$?

8soln. $\sum c_n x^{2n} = \sum c_n \left(x^2\right)^n$ converges precisely when $|x^2| < 16$, or equivalently, when $|x|^2 < 16$, or equivalently, when $|x| < \sqrt{16}$.

9. Using a known (commonly used) Taylor series, find a power series representation of the function

$$f(x) \;=\; \frac{2}{3-x}$$

about the center $x_0 = 0$ and state when this representation is valid. Hint, by simple algebra,

$$f(x) \;=\; \frac{2}{3-x} \;=\; \left(\frac{2}{3}\right)\left(\frac{1}{1-\frac{x}{3}}\right) .$$

9soln.

$$f(x) = \frac{2}{3-x} = \frac{2}{3}\left[\frac{1}{1-\left(\frac{x}{3}\right)}\right] \overset{\text{GS}}{=} \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2}{3^{n+1}}\,x^n .$$

The Geometric Series expansion (GS) is valid $\iff \left|\frac{x}{3}\right| < 1 \iff |x| < 3$

10. Using a known (commonly used) Taylor series, find a power series representation of the function

$$f(x) \;=\; \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$.
Hint. Start with the Geometric Series (Prof. Girardi sometimes called him the work moose) and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} \;=\; (-1)(1-x)^{-2}\,D_x(1-x) \;=\; (-1)(1-x)^{-2}\,(-1) \;=\; (1-x)^{-2} .$$

10soln.

Start with Geometric Series and take Derivatives as many times as need. Geometric Series is valid when $|x|<1$ so resulting power series expansions will also be valid when $|x|<1$.

Geometric Series $\Rightarrow$ $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \overset{D_x}{\Longrightarrow} (1-x)^{-2} = \sum_{k=1}^{\infty} k\,x^{k-1}$

$\overset{D_x}{\Longrightarrow} 2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1)\,x^{k-2} \overset{D_x}{\Longrightarrow} 2\cdot 3(1-x)^{-4} = \sum_{k=3}^{\infty} k(k-1)(k-2)\,x^{k-3}$

So

$$(1-x)^{-4} = \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{6}\,x^{k-3} = \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6}\,x^n$$

let $k-3=n$ $\Rightarrow k = n+3$

11. Find the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$.

11soln. The computations below show that the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$ is $P_3(x) = 5 + 3(x-1) + 9(x-1)^2 + 10(x-1)^3$.

we were given $x_0 = 1$			
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$\frac{f^{(n)}(x_0)}{n!}$
0	$x^5 - x^2 + 5$	5	$\frac{5}{0!} = \frac{5}{1} = 5$
1	$5x^4 - 2x$	$5 - 2 = 3$	$\frac{3}{1!} = \frac{3}{1} = 3$
2	$5 \cdot 4x^3 - 2$	$20 - 2 = 18$	$\frac{18}{2!} = \frac{18}{2} = 9$
3	$5 \cdot 4 \cdot 3x^2$	$(5)(4)(3)$	$\frac{(5)(4)(3)}{3!} = \frac{(5)(4)(3)}{(3)(2)} = \frac{(5)(4)}{2} = 10$

12. Find the 17th order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = 6x^7 + 5$.
Hint: We know $f(x) = P_{17}(x) + R_{17}(x)$. Think about what Taylor's (big) Remainder Theorem says about $y = R_{17}(x)$ for the function $f(x) = 6x^7 + 5$.

12soln. Note that the n^{th}-derivative of a polynomial of degree N is zero if $n > N$. So if $n \geq 8$, then $f^{(n)}(x) = 0$ for all $x \in \mathbb{R}$. Let's follow the notation from Problem **0B**.
If $N \geq 7$, then $f^{(N+1)}(c) = 0$ for any $c \in \mathbb{R}$ and so

$$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} (x-1)^{N+1} \right| = \frac{0}{(N+1)!} |x-1|^{N+1} = 0 \ ,$$

and so $P_N(x) = f(x)$. So $P_{17}(x) = f(x)$.

13. Find a parameterization for the line segment from $(-1, 2)$ to $(10, -6)$ for $0 \leq t \leq 1$.

13soln. ans: $x = -1 + 11t$ and $y = 2 - 8t$

$$\begin{aligned} x(t) &= -1 + \big(10 - ({}^{-}1)\big)\, t = -1 + 11t \\ y(t) &= 2 + \big({}^{-}6 - 2\big)\, t \qquad = 2 - 8t. \end{aligned}$$

14. Find $\dfrac{dy}{dx}$ for the parameterized curve given by

$$\begin{aligned} x &= 2t^2 + 1 \\ y &= 3t^3 + 2 \end{aligned}$$

for $1 < t < \infty$.

14soln.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t^2}{4t} = \frac{9t}{4}$$

15. Find an equation of the tangent line to the curve at the point corresponding to $t = 11\pi$.

$x = t \sin t$

$y = t \cos t.$

15soln. ans: $y = \frac{x}{11\pi} - 11\pi$

$(x(11\pi), y(11\pi)) = (0, -11\pi)$

$$\left.\frac{dy}{dx}\right|_{t=11\pi} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=11\pi} = \left.\frac{\cos t - t\sin t}{\sin t + t\cos t}\right|_{t=11\pi} = \frac{-1-0}{0-11\pi} = \frac{1}{11\pi}.$$

So equation of tangent line to curve when $t = 11\pi$ is

$$\begin{aligned}\left(y -^{-} 11\pi\right) &= \frac{1}{11\pi}(x-0)\\ y + 11\pi &= \frac{1}{11\pi}x\\ y &= \frac{1}{11\pi}x - 11\pi.\end{aligned}$$

16. Describe the motion of a puffo whose position (x, y) is parameterized by

$x = 6 \sin t$

$y = 3 \cos t$

for $0 \le t \le 2\pi$.

16soln. ans: Moves once clockwise along the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ starting and ending at $(0, 3)$.

Since $\left[\frac{x(t)}{6}\right]^2 + \left[\frac{y(t)}{3}\right]^2 = [\sin t]^2 + [\cos t]^2 = 1$,
the puffo is moving along the ellipse $\left[\frac{x}{6}\right]^2 + \left[\frac{y}{3}\right]^2 = 1$, i.e., along the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$.
He starts at $(x(0), y(0)) = (6\sin 0, 3\cos 0) = (0, 3)$.
He finishes at $(x(2\pi), y(2\pi)) = (6\sin 2\pi, 3\cos 2\pi) = (0, 3)$.
As he moves from $t = 0$ to $t = 2\pi$, he traces out the ellipse one time.
To figure out if he is moving CW or CCW, note $\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = \left(6\sin\frac{\pi}{2}, 3\cos\frac{\pi}{2}\right) = (6, 0)$.
So Mr. Puffo is moving clockwise.