MARK BOX		
PROBLEM	POINTS	
0	10	
1	10	
2	5	
3	10	
4–16	65=13x5	
%	100	

HAND IN PART

NAME: ______________________________

PIN: ______________________________

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
  You can take this part home to learn from and to check your answers once the solutions are posted.
- On Problem 0, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold during the exam (it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets.
- Cheating is grounds for a F in the course.
- At a student's request, I will project my watch upon the projector screen.
  Upon request, you will be given as much (blank) scratch paper as you need.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.7–10.10, 11.1, 11.2 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________________

0. Fill-in the boxes.

0A. Power Series. Consider a (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \,(x-x_0)^n \;, \tag{0A}$$

with radius of convergence $R \in [0,\infty]$. (Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)
Without any other further information of $\{a_n\}_{n=0}^{\infty}$, answer the following questions.
The choices for the next 4 boxes are: **AC**, **CC**, **DIVG**, or **anything**.

AC stands for: *is always absolutely convergent.*
CC stands for: *is always conditionally convergent.*
DIVG stands for: *is always divergent.*
anything stands for: *can do anything* i.e., there are examples showing that it can AC, CC, or DIVG.

0A.1. At the center $x = x_0$, the power series in (0A) [ ].

0A.2. For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series in (0A) [ ].

0A.3. For $x \in \mathbb{R}$ such that $|x - x_0| > R$, the power series in (0A) [ ].

0A.4. If $0 < R < \infty$, then for the endpoints $x = x_0 \pm R$, the power series in (0A) [ ].

0A.5. Furthermore, if α and β are in the interval $(x_0 - R, x_0 + R)$, then (Hint: note the $\Big|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta}$ already in there.)

$$\int_{x=\alpha}^{x=\beta} h(x)\,dx \;=\; \sum_{n=0}^{\infty} \boxed{} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

0B. Taylor/Maclaurin Polynomials and Series.
Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

0B.1. The n^{th} Taylor coefficient of $y = f(x)$ about x_0 is

$c_n =$ [ ]

0B.2. The N^{th}-order Taylor polynomial of $y = f(x)$ about x_0, in open form (so with $\ldots$ and without a $\sum$-sign), is

$P_N(x) =$ [ ]

0B.3. The Taylor series of $y = f(x)$ about x_0, in closed form (so, with a $\sum$-sign and without $\ldots$), is

$P_\infty(x) =$ [ ]

0C. Parametric Curves. Consider the curve $\mathcal{C}$ parameterized by

$$x = x(t)$$
$$y = y(t)$$

for $a \leq t \leq b$.

0C.1. Express $\frac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\frac{dy}{dx} =$ ☐

0C.2. The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

Arc Length = ☐

1. Commonly Used Taylor Series
Fill in Problem 1's blank boxes with the choices a – ℓ, which are provided below.
You may use a choice more than once or not at all.
A sample question is already done for you.

a. $\sum_{n=0}^{\infty} x^n$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

c. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

e. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

f. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

g. $x \in \mathbb{R}$

h. $(-1, 1)$

i. $[-1, 1]$

j. $(-1, 1]$

k. $[-1, 1)$

ℓ. none of the others

sample. A power series expansion for $y = \frac{1}{1-x}$ is [a] and is valid precisely when [h].

1.1. A power series expansion for $y = \cos x$ is ☐ and is valid precisely when ☐.

1.2. A power series expansion for $y = \sin x$ is ☐ and is valid precisely when ☐.

1.3. A power series expansion for $y = e^x$ is ☐ and is valid precisely when ☐.

1.4. A power series expansion for $y = \ln(1 + x)$ is ☐ and is valid precisely when ☐.

1.5. A power series expansion for $y = \tan^{-1} x$ is ☐ and is valid precisely when ☐.

2. Identify a parametrization of a circle with center at $(0,0)$ and radius 1, which is traced in the various below described fashions, by fill in the blanks boxes with the below choices A – I. You may use a choice more than once or not at all.

2.1. traced once, starting at $(1,0)$, in the counterclockwise direction. A parameterization is ☐

2.2. traced once, starting at $(1,0)$, in the clockwise direction. A parameterization is ☐

2.3. traced once, starting at $(-1,0)$, in the counterclockwise direction. A parameterization is ☐

2.4. traced once, starting at $(-1,0)$, in the clockwise direction. A parameterization is ☐

2.5. traced once, starting at $(0,-1)$, in the counterclockwise direction. A parameterization is ☐

$x(t) = \cos t$ $y(t) = \sin t$	for $0 \le t \le 2\pi$	(A)
$x(t) = \cos t$ $y(t) = -\sin t$	for $0 \le t \le 2\pi$	(B)
$x(t) = -\cos t$ $y(t) = \sin t$	for $0 \le t \le 2\pi$	(C)
$x(t) = -\cos t$ $y(t) = -\sin t$	for $0 \le t \le 2\pi$	(D)
$x(t) = \sin t$ $y(t) = \cos t$	for $0 \le t \le 2\pi$	(E)
$x(t) = \sin t$ $y(t) = -\cos t$	for $0 \le t \le 2\pi$	(F)
$x(t) = -\sin t$ $y(t) = \cos t$	for $0 \le t \le 2\pi$	(G)
$x(t) = -\sin t$ $y(t) = -\cos t$	for $0 \le t \le 2\pi$	(H)
None of the other choices.		(I)

3. Taylor's Remainder Theorem.

3a. Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Thus, $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Remainder Theorem tells us that, for each $x \in I$,

$R_N(x) \;=\;$ [ ] for some c between [ ] and [ ] .

3b. **In this problem, you must show your work and clearly explain your thought process.**
Using Taylor's (Big) Remainder Theorem (and not using the facts on the *Commonly Used Taylor Series* handout), show that

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \;=\; e^2 \;.$$

▷. Hint. Indeed, the facts listed on the *Commonly Used Taylor Series* are shown by using Taylor's Remainder Theorem so you can think of this problem as verifying/showing one of these facts listed on the *Commonly Used Taylor Series* handout.

▷. Hint. Consider the function $f\,(x) = e^x$. Note that then $f\,(2) = e^2$.

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
14	14a	14b	14c	14d	14e					
15	15a	15b	15c	15d	15e					
16	16a	16b	16c	16d	16e					
							5	2	0	0
							Total:			

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

4. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} c_n x^n$, which is valid in some interval $(-R, R)$ where $R > 0$.

a. Then $f(0)$ must be 0.

b. Then $f(0)$ must be c_0.

c. Then $f(0)$ must be c_1.

d. Then we know that $f(0)$ exists but we do not know what the value of $f(0)$ is.

e. None of the others.

5. Let the function $y = f(x)$ have a power series power series representation $\sum_{n=0}^{\infty} a_n x^n$, which is valid in some interval J containing 0 and the raduis of J striclty positive. Consider the two statements:
(A) If $y = f(x)$ is an even function (i.e., $f(-x) = f(x)$), then $a_1 = a_3 = a_5 = \cdots = 0$.
(B) If $y = f(x)$ is an odd function (i.e., $f(-x) = -f(x)$), then $a_0 = a_2 = a_4 = \cdots = 0$.

a. Both (A) and (B) are true.

b. Both (A) and (B) are false.

c. (A) is true but (B) is false.

d. (A) is false but (B) is true.

e. None of the others.

►. **Problems 4 and 5 were meant to help you with Problem 1. ☺☺☺**

6. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{10^n}$$

Recall that the interval of convergence is the set of x's for which the power series converges, either absolutely or conditionally.

a. $(-10, 10)$

b. $[-10, 10]$

c. $(-8, 12)$

d. $[-8, 12]$

e. None of the others.

7. What is the LARGEST interval for which the power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

is absolutely convergent?

a. $(1, 5)$

b. $(-4, -2)$

c. $(-5, -1)$

d. $[-5, -1]$

e. None of the others.

8. Suppose that the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is 16. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{2n}$?

a. 1

b. $\sqrt{16}$

c. 16

d. 16^2

e. None of the others.

9. Using a known (commonly used) Taylor series, find a power series representation of the function

$$f(x) \;=\; \frac{2}{3-x}$$

about the center $x_0 = 0$ and state when this representation is valid. Hint, by simple algebra,

$$f(x) \;=\; \frac{2}{3-x} \;=\; \left(\frac{2}{3}\right)\left(\frac{1}{1-\frac{x}{3}}\right).$$

a. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n} x^n$, valid for $|x| < 1$

b. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}\, x^n$, valid for $|x| < 3$

c. $\displaystyle\sum_{n=0}^{\infty}\frac{2}{3^{n+1}}\, x^n$, valid for $|x| < 1$

d. $\displaystyle\sum_{n=0}^{\infty}\frac{2}{3^{n+1}}\, x^n$, valid for $|x| < 3$

e. None of the others.

10. Using a known (commonly used) Taylor series, find a power series representation of the function

$$f(x) \;=\; \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$.
Hint. Start with the Geometric Series (Prof. Girardi sometimes called him the work moose) and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} \;=\; (-1)(1-x)^{-2}\, D_x(1-x) \;=\; (-1)(1-x)^{-2}\,(-1) \;=\; (1-x)^{-2}\,.$$

a. $\displaystyle\sum_{n=0}^{\infty}(n)(n-1)(n-2)\,x^n$

b. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\,(n)(n-1)(n-2)\,x^n$

c. $\displaystyle\sum_{n=0}^{\infty}\frac{(n+3)(n+2)(n+1)}{6}\,x^n$

d. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\,\frac{(n+3)(n+2)(n+1)}{6}\,x^n$

e. None of the others.

11. Find the 3[rd] order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$.

a. $P_3(x) = 5 + 3(x-1) + 9(x-1)^2 + 10(x-1)^3$

b. $P_3(x) = 5 + 3(x-1) + 18(x-1)^2 + 60(x-1)^3$

c. $P_3(x) = 5 + 3x + 9x^2 + 10x^3$

d. $P_3(x) = 5 + 3x + 18x^2 + 60x^3$

e. None of the others.

12. Find the 17[th] order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = 6x^7 + 5$. Hint: We know $f(x) = P_{17}(x) + R_{17}(x)$. Think about what Taylor's (big) Remainder Theorem says about $y = R_{17}(x)$ for the function $f(x) = 6x^7 + 5$.

a. $P_{17}(x) = 5 + 6(x+1)^7$.

b. $P_{17}(x) = 5 + 6(x-1)^7$.

c. $P_{17}(x) = 5 + 6x^7$.

d. It does not exist.

e. None of the others.

13. Find a parameterization for the line segment from $(-1, 2)$ to $(10, -6)$ for $0 \le t \le 1$.

a. $x = 10 - 8t$ and $y = -1 + t$

b. $x = -1 + 11t$ and $y = 2 - 8t$

c. $x = -1 + 11t$ and $y = -6 - 8t$

d. $x = -1 - 11t$ and $y = -8t$

e. None of the others.

14. Find $\dfrac{dy}{dx}$ for the parameterized curve given by

$$\begin{aligned} x &= 2t^2 + 1 \\ y &= 3t^3 + 2 \end{aligned}$$

for $1 < t < \infty$.

a. $\dfrac{dy}{dx} = \dfrac{4}{9t}$.

b. $\dfrac{dy}{dx} = \dfrac{9t}{4}$.

c. $\dfrac{dy}{dx} = 4t$.

d. $\dfrac{dy}{dx} = 9t^2$.

e. None of the others.

15. Find an equation of the tangent line to the curve at the point corresponding to $t = 11\pi$.

$x = t \sin t$

$y = t \cos t.$

a. $y = \dfrac{x}{11\pi} + 12\pi$

b. $y = \dfrac{x}{11\pi} - 11\pi$

c. $y = \dfrac{x}{11\pi} + 11\pi$

d. $y = \dfrac{x}{11\pi} - 12\pi$

e. None of the others.

16. Describe the motion of a puffo whose position (x, y) is parameterized by

$x = 6 \sin t$

$y = 3 \cos t$

for $0 \le t \le 2\pi$.

a. Moves once counterclockwise along the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ starting and ending at $(6, 0)$.

b. Moves once counterclockwise along the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$ starting and ending at $(-3, 0)$.

c. Moves once clockwise along the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ starting and ending at $(0, 3)$.

d. Moves once clockwise along the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ starting and ending at $(-3, 0)$.

e. None of the others.