MARK BOX		
PROBLEM	POINTS	
0	10	
1	15	
2	10	
3	10	
4	10	
5	10	
6-12	35=7x5	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
  You can take this part home to learn from and to check your answers once the solutions are posted.
- On Problem 0, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, personal notes, electronic devices, and any device with which you can connect to the internet. Unauthorized materials (including cell phones, as well as your watch) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold during the exam (it will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets.
- Cheating is grounds for a F in the course.
- At a student's request, I will project my watch upon the projector screen.
  Upon request, you will be given as much (blank) scratch paper as you need.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- This exam covers (from *Calculus* by Thomas, 13th ed., ET): §10.1–10.6 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0.1. For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums, so $s_n = \sum_{k=1}^{n} a_k$. By definition, the formal series $\sum a_n$ converges if and only if

the $\lim_{n\to\infty} s_n$ converges (to a finite number). [also ok: the $\lim_{n\to\infty} s_n$ exists (in $\mathbb{R}$)]

0.2. **Geometric Series**. Fill in the boxes with the proper range of $r \in \mathbb{R}$.

- The series $\sum r^n$ converges if and only if r satisfies $\boxed{|r| < 1}$.

0.3. ***p*-series**. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if $\boxed{p > 1}$.

0.4. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If $\boxed{0 \leq a_n \leq c_n}$ (only $a_n \leq c_n$ is also ok b/c given $a_n \geq 0$) when $n \geq 17$ and $\sum c_n$ converges, then $\sum a_n$ converges.
- If $\boxed{0 \leq d_n \leq a_n}$ (need $0 \leq d_n$ part here) when $n \geq 17$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.5. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $\boxed{0 < L < \infty}$, then $[\sum b_n$ converges $\iff$ $\sum a_n$ converges $]$.
- If $L = 0$, then $\boxed{[\sum b_n \text{ converges} \implies \sum a_n \text{ converges}]}$.
- If $L = \infty$, then $\boxed{[\sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}]}$.

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.6. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

- If $\rho > 1$, then $\sum a_n$ $\boxed{\text{diverges}}$.
- If $\rho < 1$, then $\sum a_n$ $\boxed{\text{converges absolutely}}$.

1. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true.

The symbol $\sum$ is understood to mean $\displaystyle\sum_{n=1}^{\infty}$.

Scoring this problem: A problem with precisely one answer marked and the answer is correct, 1 point. All other cases, 0 points.

<table>
<tr><td colspan="4">On the next 3: think of the n^{th}-term test for divergence and what if $a_n = \frac{1}{n}$.</td></tr>
<tr><td>1.1</td><td>[✓ T]</td><td></td><td>If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.</td></tr>
<tr><td>1.2</td><td>[✓ T]</td><td></td><td>If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.</td></tr>
<tr><td>1.3</td><td></td><td>[✓ F]</td><td>If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.</td></tr>
<tr><td colspan="4">On the next 5: think of def. of AC/CC/Divergent, the Big Theorem AC ⇒ convergence, and $\sum \frac{(-1)^n}{n}$ is CC.</td></tr>
<tr><td>1.4</td><td>[✓ T]</td><td></td><td>A series $\sum a_n$ is precisely one of the following:
absolutely convergent, conditionally convergent, divergent.</td></tr>
<tr><td>1.5</td><td>[✓ T]</td><td></td><td>If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ is either absolutely convergent or divergent.</td></tr>
<tr><td>1.6</td><td>[✓ T]</td><td></td><td>If $\sum |a_n|$ converges, then $\sum a_n$ converges.</td></tr>
<tr><td>1.7</td><td>[✓ T]</td><td></td><td>If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.</td></tr>
<tr><td>1.8</td><td></td><td>[✓ F]</td><td>If $\sum |a_n|$ diverges, then $\sum a_n$ diverges.</td></tr>
<tr><td colspan="4">On the next one: think of a Theorem from class and what if $b_n = -a_n$.</td></tr>
<tr><td>1.9</td><td></td><td>[✓ F]</td><td>If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.</td></tr>
<tr><td colspan="4">On the next 2: think of a Theorem from class and what if $f(x) = \sin(\pi x)$.</td></tr>
<tr><td>1.10</td><td>[✓ T]</td><td></td><td>If a function $f\colon [0,\infty) \to \mathbb{R}$ satisfies that $\lim_{x\to\infty} f(x) = L$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{n\to\infty} a_n = L$.</td></tr>
<tr><td>1.11</td><td></td><td>[✓ F]</td><td>If a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies that $\lim_{n\to\infty} a_n = L$ and $f\colon [0,\infty) \to \mathbb{R}$ is a function satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{x\to\infty} f(x) = L$.</td></tr>
<tr><td colspan="4">The next 4 are from a MML homework problem.</td></tr>
<tr><td>1.12</td><td>[✓ T]</td><td></td><td>The $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=17}^{\infty} a_n$ converges.</td></tr>
<tr><td>1.13</td><td></td><td>[✓ F]</td><td>If $\sum a_n$ converges, then $\sum (a_n + 0.01)$ converges.</td></tr>
<tr><td>1.14</td><td></td><td>[✓ F]</td><td>If $\sum r^n$ diverges, then $\sum (r + 0.01)^n$ diverges, for any fixed real number r.</td></tr>
<tr><td>1.15</td><td></td><td>[✓ F]</td><td>If $\sum \frac{1}{n^p}$ converges, then $\sum \frac{1}{n^{p-0.01}}$ converges, for any fixed real number p.</td></tr>
</table>

2. Geometric Series. (On this page, you should do basic algebra but you do NOT have to do any grade-school arithmetic (eg, you can leave $(\frac{17}{18})^{19}$ as just that.) Let, for $N \geq 17$,

$$s_N \;=\; \sum_{n=17}^{N} 3\left(\frac{5}{7}\right)^n \,.$$

2.1. Below the box, using the method from class (rather than some formula), find an expression for s_N without using some dots … nor a summation $\sum$ sign. Then put your answer in the box.

$$\boxed{s_N \;=\; \left(\frac{7}{2}\right)(3)\left[\left(\frac{5}{7}\right)^{17} - \left(\frac{5}{7}\right)^{N+1}\right] \qquad \text{(answers will vary)}}$$

Find an expression for $s_N - rs_N$, which results in a *cancellation heaven.*

$$\begin{array}{rcl} s_N & = 3\Big[& \left(\frac{5}{7}\right)^{17} + \cancel{\left(\frac{5}{7}\right)^{18}} + \ldots + \cancel{\left(\frac{5}{7}\right)^{N-1}} + \cancel{\left(\frac{5}{7}\right)^{N}} \Big] \\ & & \qquad\quad \swarrow \qquad\qquad \cdot\swarrow \qquad\qquad \swarrow \\ \left(\frac{5}{7}\right) s_N & = 3\Big[& \cancel{\left(\frac{5}{7}\right)^{18}} + \cancel{\left(\frac{5}{7}\right)^{19}} + \ldots + \cancel{\left(\frac{5}{7}\right)^{N}} + \left(\frac{5}{7}\right)^{N+1} \Big] \end{array}$$

substract

$$\left(1-\left(\frac{5}{7}\right)\right)s_N \overset{\text{Ⓐ}}{=} s_N - \left(\frac{5}{7}\right)s_N = 3\left[\left(\frac{5}{7}\right)^{17} - \left(\frac{5}{7}\right)^{N+1}\right]$$

and so

$$s_N \overset{\text{Ⓐ}}{=} \frac{3\left[\left(\frac{5}{7}\right)^{17} - \left(\frac{5}{7}\right)^{N+1}\right]}{1-\left(\frac{5}{7}\right)} \overset{\text{Ⓐ}}{=} \frac{3\left[\left(\frac{5}{7}\right)^{17} - \left(\frac{5}{7}\right)^{N+1}\right]}{\frac{2}{7}} \overset{\text{Ⓐ}}{=} \boxed{\left(\frac{7}{2}\right)(3)\left[\left(\frac{5}{7}\right)^{17} - \left(\frac{5}{7}\right)^{N+1}\right]}$$

2.2. If $\sum_{n=17}^{\infty} 3\left(\frac{5}{7}\right)^n$ diverges, then write diverges in the box. If $\sum_{n=17}^{\infty} 3\left(\frac{5}{7}\right)^n$ converges, write the number it converges to in the box. Justify your answer below the box.

$$\boxed{\sum_{n=17}^{\infty} 3\left(\frac{5}{7}\right)^n = \left(\frac{7}{2}\right)(3)\left(\frac{5}{7}\right)^{17}}$$

The series is a geometric series with ratio $r = \dfrac{5}{7}$; hence, since $|r| < 1$, we know the series converges.
The series will converge to $\lim_{N\to\infty} s_N$.
Since $\left|\frac{5}{7}\right| < 1$, the $\displaystyle\lim_{n\to\infty} \frac{5}{7} = 0$ and so

$$\lim_{N\to\infty} s_n = \lim_{N\to\infty}\left(\frac{7}{2}\right)(3)\left[\left(\frac{5}{7}\right)^{17} - \left(\frac{5}{7}\right)^{N+1}\right] \;=\; \left(\frac{7}{2}\right)(3)\left[\left(\frac{5}{7}\right)^{17} - 0\right]$$

3. Below the choice-boxes (AC/CC/Divg), carefully justify the given series's behavior. Be sure to specify which test(s) you are using and clearly explain your logic. Then check the correct choice-box.

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

☒ absolutely convergent

☐ ~~conditionally convergent~~ (cannot be since it's a positive termed series)

☐ divergent

Thinking Land.

$$\frac{1}{n^2-1} \overset{n \text{ big}}{\approx} \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ (p-series, $p = 2 > 1$ so) converges. So we try a comparison test. Note for each $n \in \mathbb{N}$

$$\frac{1}{n^2} \le \frac{1}{n^2-1}$$

so if use DCT, comparing the given series with the convergent p-series $\sum \frac{1}{n^2}$, would be boundeding below by a convergent, NO GO. So let's try LCT, comparing the given series with $\sum \frac{1}{n^2}$.
End of Thinking Land.

Let

$$a_n = \frac{1}{n^2-1} \qquad \text{and} \qquad b_n = \frac{1}{n^2}.$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{n^2}{n^2-1} \\
&= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2-1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^2}} \\
&= \frac{1}{1-0} \\
&= 1.
\end{aligned}$$

By the LCT (Limit Comparison Test) since $0 < 1 < \infty$,

the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ *do the same thing.*

The series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ (it's a p-series with $p = 2 > 1$) converges.

So the $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ also converges.

4. For a natural number $n > 2$, let

$$a_n = \frac{(n-1)!}{(3n)!}$$

4.1. Below the box, find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it. Then put your answer in the box.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{n}{(3n+1)\,(3n+2)\,(3n+3)}}$$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{((n+1)-1)!}{(3\,(n+1))!}\;\frac{(3n)!}{(n-1)!} = \frac{n!}{(n-1)!}\;\frac{(3n)!}{(3n+3)!} \\ &= \frac{(n-1)!n}{(n-1)!}\;\frac{(3n)!}{(3n)!\,(3n+1)\,(3n+2)\,(3n+3)} = \frac{n}{(3n+1)\,(3n+2)\,(3n+3)} \end{aligned}$$

4.2. Below the choice-boxes (AC/CC/Divg), carefully justify the given series's behavior. Be sure to specify which test(s) you are using and clearly explain your logic. Then check the correct choice-box.You may use previous parts of this problem.

$$\sum_{n=2}^{\infty} \frac{(n-1)!}{(3n)!}$$

☒ absolutely convergent

☐ ~~conditionally convergent~~ (cannot be since it's a positive termed series)

☐ divergent

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \overset{\text{from above part}}{=} \lim_{n\to\infty} \frac{n}{(3n+1)\,(3n+2)\,(3n+3)} \\ &\overset{\textcircled{A}}{=} \lim_{n\to\infty} \end{aligned}$$

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &\underset{\text{part}}{\overset{\text{previous}}{=}} \frac{n}{(3n+1)\,(3n+2)\,(3n+3)} \overset{\textcircled{A}}{=} \frac{\frac{n}{n^3}}{\frac{(3n+1)}{n}\,\frac{(3n+2)}{n}\,\frac{(3n+3)}{n}} \\ &\overset{\textcircled{A}}{=} \frac{\frac{1}{n^2}}{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)} \xrightarrow{n\to\infty} \frac{0}{(3)\,(3)\,(3)} = 0. \end{aligned}$$

Since $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, by the ratio test, the series $\sum a_n$ is absolutely convergent.

5. Below the choice-boxes (AC/CC/Divg), carefully justify the given series's behavior. Be sure to specify which test(s) you are using and clearly explain your logic. Then check the correct choice-box.

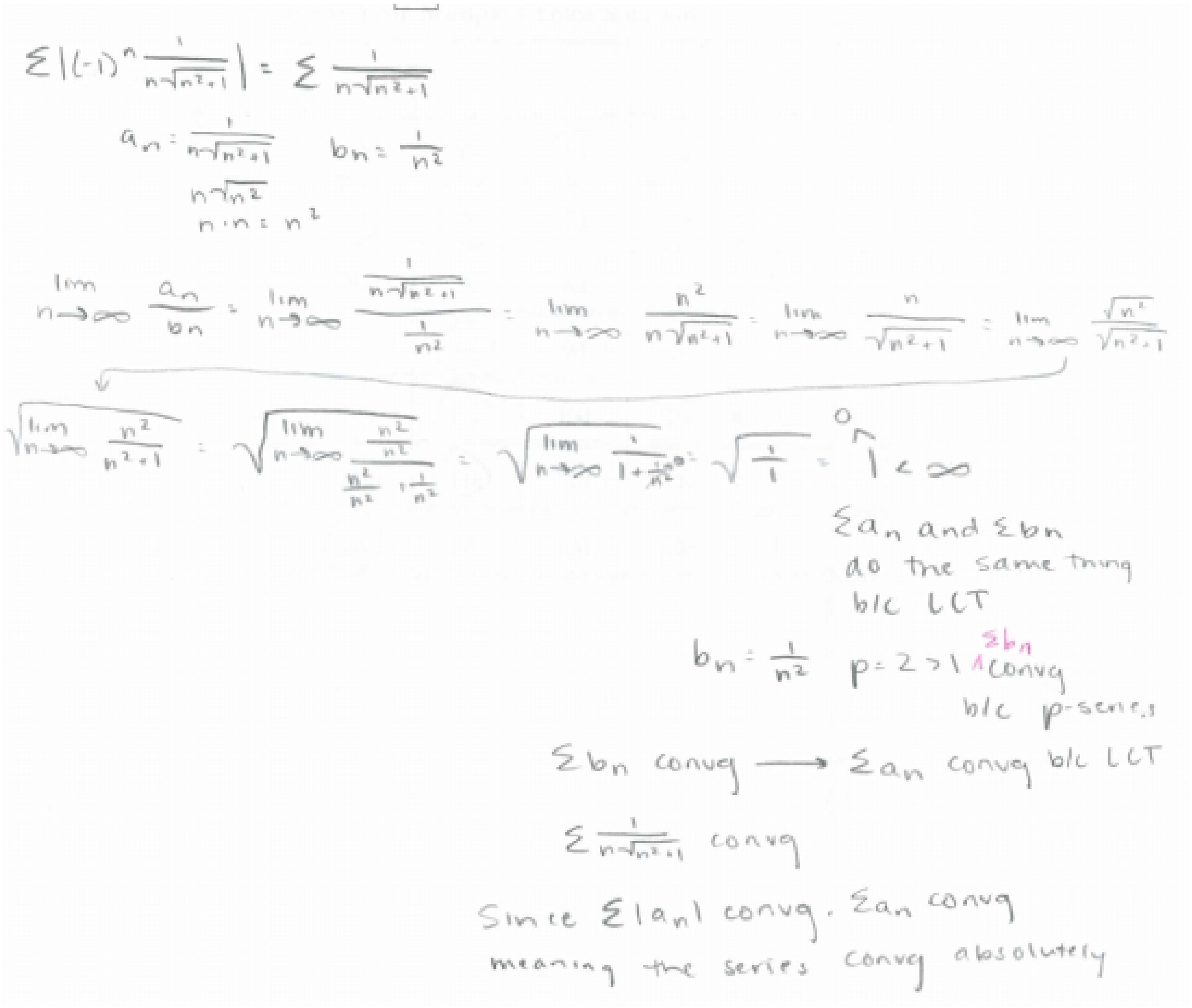
$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\sqrt{n^2+1}}$$

☒ absolutely convergent

☐ conditionally convergent

☐ divergent

This was is number 4 from the 15 Serious Series Problems.



$\Sigma \left| (-1)^n \frac{1}{n\sqrt{n^2+1}} \right| = \Sigma \frac{1}{n\sqrt{n^2+1}}$

$a_n = \frac{1}{n\sqrt{n^2+1}}$ $b_n = \frac{1}{n^2}$

$n\sqrt{n^2}$

$n \cdot n = n^2$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2+1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2+1}}$

$\sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = \sqrt{\lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}}} = \sqrt{\lim_{n\to\infty} \frac{1}{1+\overset{0}{\frac{1}{n^2}}}} = \sqrt{\frac{1}{1}} = 1 < \infty$

Σa_n and Σb_n do the same thing b/c LCT

$b_n = \frac{1}{n^2}$ $p = 2 > 1$ Σb_n convg b/c p-series

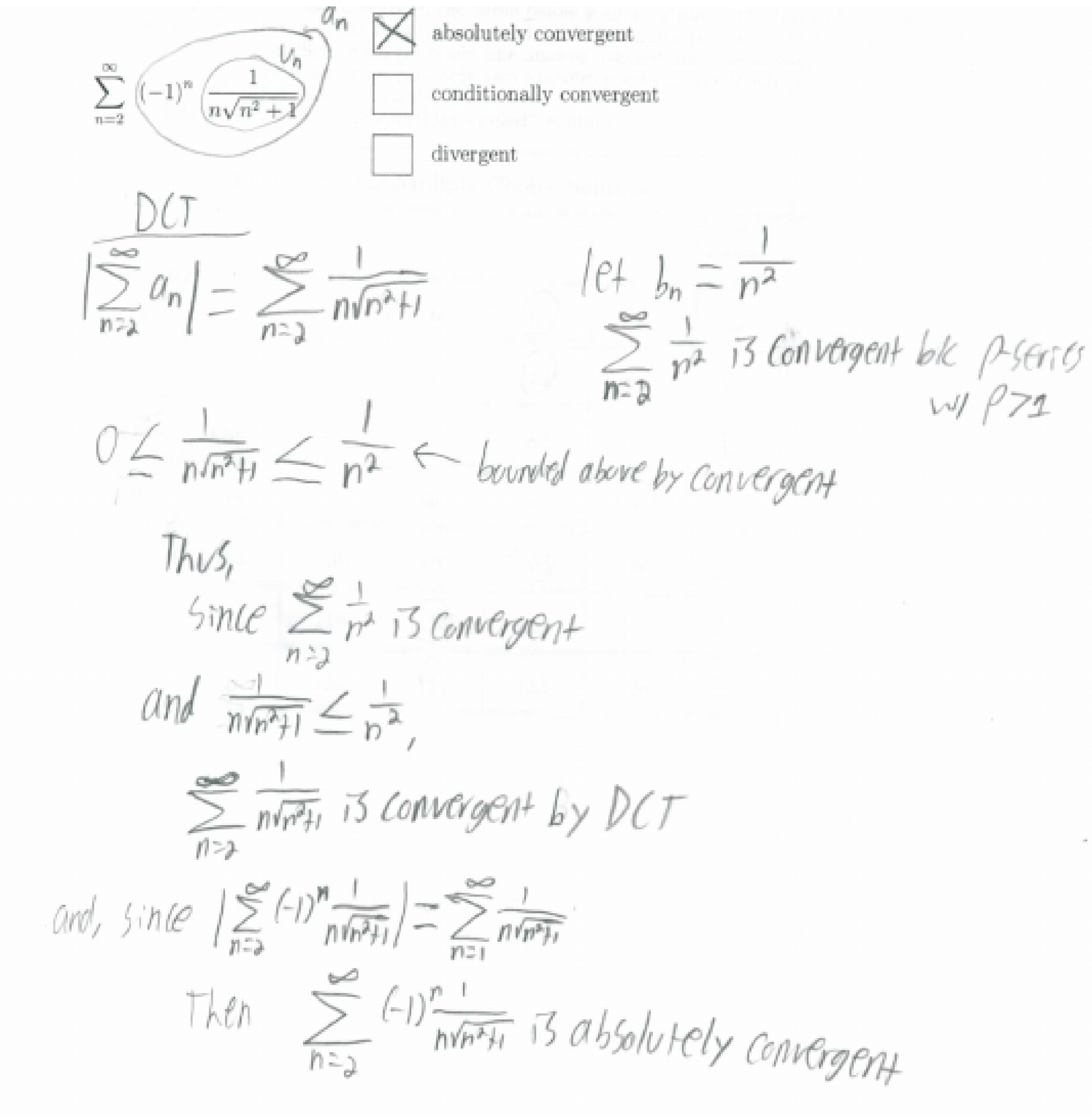
Σb_n convg $\longrightarrow$ Σa_n convg b/c LCT

$\Sigma \frac{1}{n\sqrt{n^2+1}}$ convg

Since $\Sigma |a_n|$ convg. Σa_n convg meaning the series convg absolutely

Over for another sample student solution.

5.soln. Another sample student solution.



$$\sum_{n=2}^{\infty} \overset{a_n}{(-1)^n \overset{v_n}{\frac{1}{n\sqrt{n^2+1}}}}$$

☒ absolutely convergent

☐ conditionally convergent

☐ divergent

DCT

$\left|\sum_{n=2}^{\infty} a_n\right| = \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

let $b_n = \frac{1}{n^2}$

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent b/c p-series w/ $p>1$

$0 \le \frac{1}{n\sqrt{n^2+1}} \le \frac{1}{n^2}$ ← bounded above by convergent

Thus,

since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent

and $\frac{1}{n\sqrt{n^2+1}} \le \frac{1}{n^2}$,

$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ is convergent by DCT

and, since $\left|\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\sqrt{n^2+1}}\right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

Then $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\sqrt{n^2+1}}$ is absolutely convergent

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
6	6a	6b	6c	[✓ 6d]	6e					
7	7a	7b	7c	[✓ 7d]	7e					
8	[✓ 8a]	8b	8c	8d	8e					
9	9a	9b	[✓ 9c]	9d	9e					
10	10a	10b	[✓ 10c]	10d	10e					
11	11a	11b	[✓ 11c]	11d	11e					
12	12a	[✓ 12b]	12c	12d	12e					
							5	2	0	0
							Total:			

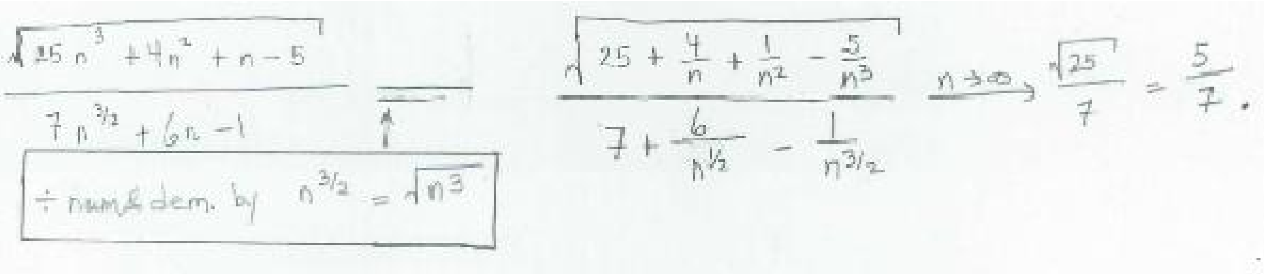
STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

Abbreviations used:
- DCT is Direct Comparison Test.
- LCT is Limit Comparison Test.
- AST is Alternating Series Test.

6. Limit of a sequence. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{25n^3+4n^2+n-5}}{7n^{\frac{3}{2}}+6n-1}\ .$$

6soln.



$$\frac{\sqrt{25n^3+4n^2+n-5}}{7n^{3/2}+6n-1} \qquad \frac{\sqrt{25+\frac{4}{n}+\frac{1}{n^2}-\frac{5}{n^3}}}{7+\frac{6}{n^{1/2}}-\frac{1}{n^{3/2}}} \xrightarrow{n\to\infty} \frac{\sqrt{25}}{7} = \frac{5}{7}.$$

$$\boxed{\div \text{ num \& dem. by } n^{3/2} = \sqrt{n^3}}$$

7. Consider the formal series

$$\sum_{n=1}^{\infty} \frac{1}{n+3^n}.$$

7soln. This is the first problem from the 38 Serious Series Problems.

$\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ for all $n \geq 1$. $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent geometric series $\left[|r| = \frac{1}{3} < 1\right]$, so $\sum_{n=1}^{\infty}\frac{1}{n+3^n}$ converges by the Comparison Test.

☹Note that $\frac{1}{n+3^n} \neq \frac{1}{n} + \frac{1}{3^n}$ ... unfortunately, some folks told me they were equal. ☹

8. Consider the formal series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+2)(n+7)}\ .$$

8soln. $\frac{1}{(n+2)(n+7)} \overset{n \text{ big}}{\sim} \frac{1}{(n)(n)} = \frac{1}{n^2}$. So let $b_n = \frac{1}{n^2}$ and $a_n = \frac{(-1)^n}{(n+2)(n+7)}$. Then

$$\lim_{n\to\infty}\frac{|a_n|}{b_n} = \lim_{n\to\infty}\frac{1}{(n+2)(n+7)}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{n^2}{(n+2)(n+7)}$$

$$= \lim_{n\to\infty}\frac{\frac{n^2}{n^2}}{\left(\frac{n+2}{n}\right)\left(\frac{n+7}{n}\right)} \lim_{n\to\infty}\frac{1}{\left(1+\frac{2}{n}\right)\left(1+\frac{7}{n}\right)} = \frac{1}{(1+0)\,(1+0)} = 1\ .$$

Since $0 < \lim_{n\to\infty}\frac{|a_n|}{b_n} < \infty$, by the LCT, $\sum b_n$ and $\sum |a_n|$ *do the same thing.*
We know $\sum b_n = \sum \frac{1}{n^2}$ (p-series, $p = 2 > 1$ so) converges.
So $\sum |a_n|$ converges. So $\sum a_n$ is absolutely convergent.

9. By using the Limit Comparison Test, one can show that the formal series

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}}\ . \tag{9}$$

is:

9soln. Let

$$a_n \ = \ \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}}$$

For n sufficiently big,

$$a_n \ = \ \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}} \ \overset{\text{when } n \text{ is big}}{\approx} \ \frac{n}{\sqrt{(n)\,(n)\,(n)\,(n)}} \ = \ \frac{n^1}{n^{4/2}} \ = \ \frac{1}{n}\ .$$

So we let $b_n = \frac{1}{n}$ and compute

$$\begin{aligned} \frac{a_n}{b_n} \ &= \ \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)}}\ \frac{n}{1} \ = \ \frac{n^2}{[\,(n+1)\,(n+2)\,(n+3)\,(n+4)]^{1/2}} \\ &= \ \left[\frac{n^4}{(n+1)\,(n+2)\,(n+3)\,(n+4)}\right]^{1/2} \ = \ \left[\frac{n}{(n+1)}\ \frac{n}{(n+2)}\ \frac{n}{(n+3)}\ \frac{n}{(n+4)}\right]^{1/2} \\ &\xrightarrow{n\to\infty} [\,(1)\ (1)\ (1)\ (1)\,]^{1/2} \ = \ 1\ . \end{aligned}$$

Since $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, the LCT says the $\sum a_n$ and $\sum b_n$ do the same thing. Since $\sum b_n$ is a p-series with $p = 1 \le 1$, the $\sum b_n$ diverges. So the $\sum a_n$ diverges.

10. Find **all** real numbers r satisfying that

$$\sum_{n=2}^{\infty} r^n \ = \ \frac{1}{6}\ .$$

10soln. Soln: $\frac{-1}{2}$ and $\frac{1}{3}$

First note that for the series $\sum_{n=2}^{\infty} r^n$ to converge (so that the problem even makes sense), we need

$$|r| < 1.$$

So let $\boxed{|r| < 1}$. Next, to find the sum $\sum_{n=2}^{\infty} r^n$, consider the partial sums $s_n \overset{\text{def}}{=} r^2 + r^3 + \ldots + r^{n-1} + r^n$.

Cancellation Heaven occurs with a geometric series when one computes $s_n - rs_n$. Let's see why.

$$\begin{aligned} s_n &= r^2 \ + \ r^3 + \ \ldots \ + r^{n-1} \ + \ r^n \\ r\,s_n \quad &= r^3 \ + \ r^4 + \ \ldots \ + \ r^n \quad + \ r^{n+1} \end{aligned}$$

Do you see the cancellation that would occur if we take $s_n - rs_n$?

$$\begin{array}{lcl} s_n & = & r^2 + \cancel{r^3} + \ldots + \cancel{r^{n-1}} + \cancel{r^n} \\ & & \qquad \swarrow \qquad \cdot\swarrow\cdot \qquad \swarrow \\ r\,s_n & = & \cancel{r^3} + \cancel{r^4} + \ldots + \cancel{r^n} \quad + r^{n+1} \end{array}$$

substract ———————————————————————————

$$(1-r)\ s_n \ \overset{Ⓐ}{=} \ s_n - rs_n \ = \ r^2 \qquad\qquad - r^{n+1}$$

and since $r \neq 1$, then

$$s_n \; = \; \frac{r^2 - r^{n+1}}{1-r} \qquad \xrightarrow{\text{since } |r|<1} \qquad \frac{r^2}{1-r} = \sum_{n=2}^{\infty} r^n .$$

So we are looking for $r \in \mathbb{R}$ so that $|r| < 1$ and $\frac{r^2}{1-r} = \frac{1}{6}$. Note $\left[\frac{r^2}{1-r} = \frac{1}{6}\right] \Leftrightarrow [6r^2 = 1 - r] \Leftrightarrow [6r^2 + r - 1 = 0] \Leftrightarrow$

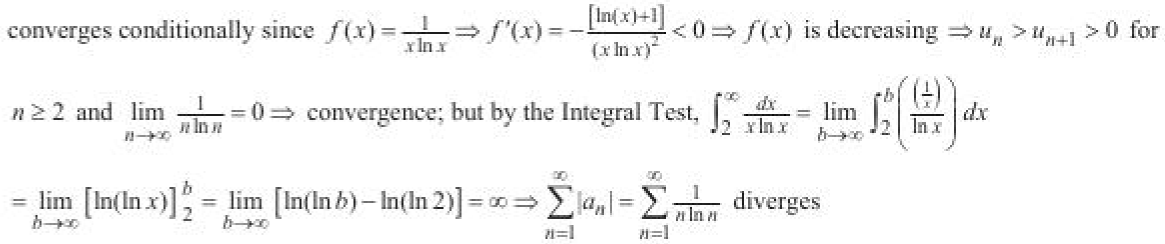
$$r = \frac{-1 \pm \sqrt{1 + 4\,(6)}}{2\,(6)} = \frac{-1 \pm 5}{12} = \begin{cases} \frac{-1+5}{12} = \frac{4}{12} = \frac{1}{3} \\ \frac{-1-5}{12} = -\frac{6}{12} = -\frac{1}{2}. \end{cases}$$

11. The series

$$\sum_{n=2}^{\infty} (-1)^{n+1} \, \frac{1}{n \ln n},$$

is

11soln.



converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x)+1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim\limits_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test, $\int_2^{\infty} \frac{dx}{x \ln x} = \lim\limits_{b\to\infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x} \right) dx$ $= \lim\limits_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \lim\limits_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty \Rightarrow \sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges

12. What is the smallest integer N such that the Alternating Series Estimate/Remainder Theorem guarentees that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \leq 0.05?$$

Note that $0.05 = \frac{0.05}{1.0000} = \frac{5}{100} = \frac{1}{20}$.

12soln. Note that $0 \leq \frac{1}{n^2} \searrow 0$ so the AST applies and tells us that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges and that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \leq \frac{1}{(N+1)^2}.$$

So

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \right| \overset{\text{have}}{\leq} \frac{1}{(N+1)^2} \overset{\text{want}}{\leq} \frac{1}{20}.$$

Note

$$\left[\frac{1}{(N+1)^2} \leq \frac{1}{20} \right] \Leftrightarrow \left[\, 20 \leq (N+1)^2 \, \right].$$

If $N = 3$, then $(N+1)^2 = (3+1)^2 = 4^2 = 16 < 20$.
If $N = 4$, the $(N+1)^2 = (4+1)^2 = 5^2 = 25 \geq 20$.