MARK BOX		
PROBLEM	POINTS	
0A	11	
0B	9	
0C	12	
0D	3	
Total for 0	35	
1-11	55= 11x5	
12	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND-IN PART. Hand-in only this part.
  (2) NOT TO HAND-IN PART. This part will not be collected. Take this part home to learn from and to check your answers when the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For problems > 11**, to receive credit you **MUST**:
  (1) **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  (2) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (3) if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems (check that you have them all) along with their points.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.7-10.10 and 11.1–11.2 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

0A. Power Series Condsider a (formal) power series

$$h(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n , \tag{0.1}$$

with radius of convergence $R \in [0, \infty]$.

(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

- For this part, answer the 4 questions by circling one and only one choice. Abbreviations: AC for absolutely convergent, CC for conditionally convergent, and DV for divergent.

  (1) For $x = x_0$, the power series $h(x)$ in (0.1)

  ⓐ is always AC b. is always CC c. is always DV

  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

  (2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series $h(x)$ in (0.1)

  ⓐ is always AC b. is always CC c. is always DV

  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

  (3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$ the power series $h(x)$ in (0.1)

  a. is always AC b. is always CC ⓒ is always DV

  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

  (4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series $h(x)$ in (0.1)

  a. is always AC b. is always CC c. is always DV

  ⓓ can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

- For this part, fill in the 7 boxes.

  Let $R > 0$ and donsider the function $y = h(x)$ defined by the power series in (0.1).

  (1) The function $y = h(x)$ is <u>always differentiable</u> on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

$$h'(x) = \sum_{n=\boxed{1}}^{\infty} \boxed{n\, a_n (x - x_0)^{n-1}} . \tag{0.2}$$

  What can you say about the radius of convergence of the power series in (0.2)?

The power series in (0.2) has the same raduis of convergence as the power series in (0.1).

  (2) The function $y = h(x)$ <u>always has an antiderivative</u> on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\, dx = \sum_{n=\boxed{0}}^{\infty} \boxed{\frac{a_n}{n+1} (x - x_0)^{n+1}} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

0B. Taylor/Maclaurin Polynomials and Series.

For this part, fill-in the 9 boxes.

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. The formula for c_n is

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

b. In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N$$

c. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n$$

d. In open form (i.e., with ... and without a $\sum$-sign)

$$P_\infty(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots$$

e. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n$$

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)}$$ for some c between x and x_0.

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = 0$.

0C. Commonly Used Taylor Series

For this part, fill in the 12 below blanks boxes with the choices a – ℓ.

(1) A power series expansion for $y = \cos x$ is [d] and is valid precisely when [g].

(2) A power series expansion for $y = \sin x$ is [e] and is valid precisely when [g].

(3) A power series expansion for $y = e^x$ is [b] and is valid precisely when [g].

(4) A power series expansion for $y = \frac{1}{1-x}$ is [a] and is valid precisely when [h].

(5) A power series expansion for $y = \ln(1+x)$ is [c] and is valid precisely when [j].

(6) A power series expansion for $y = \tan^{-1} x$ is [f] and is valid precisely when [i].

The choices a-ℓ for above 12 blank boxes. You may use a choice more then once or not at all.

a. $\sum_{n=0}^{\infty} x^n$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

c. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

e. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

f. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

g. $x \in \mathbb{R}$

h. $(-1, 1)$

i. $[-1, 1]$

j. $(-1, 1]$

k. $[-1, 1)$

ℓ. none of the others

0D. Parametric Curves In this part, fill in the 3 boxes. Consider the curve $\mathcal{C}$ parameterized by

$$x = x(t)$$
$$y = y(t)$$

for $a \le t \le b$.

1) Express $\frac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\frac{dy}{dx} = \boxed{\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$

2) The tangent line to $\mathcal{C}$ when $t = t_0$ is $y = mx + b$ where m is $\boxed{\frac{dy}{dx}}$ evaluated at $t = t_0$.

3) The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

$$\text{Arc Length} = \boxed{\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt}$$

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ For a problem with nothing marked (i.e., left blank) 1 point.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions — Do Not Write Below

PROBLEM						number of solutions circled	1	2	B	x
1	1a	(1b)	1c	1d	1e					
2	2a	2b	(2c)	2d	2e					
3	3a	3b	3c	(3d)	3e					
4	4a	4b	(4c)	4d	4e					
5	5a	(5b)	5c	5d	5e					
6	6a	6b	(6c)	6d	6e					
7	7a	7b	(7c)	7d	7e					
8	8a	(8b)	8c	8d	8e					
9	(9a)	9b	9c	9d	9e					
10	10a	(10b)	10c	10d	10e					
11	11a	(11b)	11c	11d	11e					

5	2	1	0

12. Clearly justify your solution **below** the boxes and then put your solution **in** the boxes.

Find a power series representation, centered about $x_0 = 0$, for

$$f(x) = \frac{x^2}{(1+x)^3} \tag{12.1}$$

and say when it is valid.

Express your series in **CLOSED** form (i.e., with a $\sum$-sign and without $\ldots$).

Soln: $\dfrac{x^2}{(1+x)^3} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2} x^{n+2} \overset{\text{also}}{\underset{\text{correct}}{=}} \sum_{n=2}^{\infty} (-1)^n \frac{(n)(n-1)}{2} x^n}$, valid when $\boxed{|x| < 1}$ also correct $x \in (-1, 1)$

Justification:

$x^2(1+x)^{-3}$

$f(x) = \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{(1-(-x))^3}$

$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \overset{dx}{\Rightarrow} (1-x)^{-2} = \sum_{k=1}^{\infty} k x^{k-1} \overset{dx}{\Rightarrow}$ ~~$2(1-x)^{-3}$~~

$2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1)x^{k-2} \overset{(A)}{\Rightarrow} (1-x)^{-3} = \sum_{k=2}^{\infty} \frac{k(k-1)x^{k-2}}{2}$

Given this: $f(x) = x^2 \cdot \frac{1}{(1-(-x))^3} = x^2 \sum_{k=2}^{\infty} \frac{k(k-1)(-x)^{k-2}}{2}$

Valid when: $|-x| < 1$ so $-1 < x < 1$

Now take $n = k-2$) $= x^2 \sum_{n=0}^{\infty} \frac{(n+2)(n+1)(-x)^n}{2}$ or

$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1) x^2 x^n}{2}$ or $\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1) x^{n+2}}{2}$

NOT TO HAND-IN PART
STATEMENT OF MULTIPLE CHOICE PROBLEMS

1. Find the 4^{th} order Maclaurin polynonial for

$$f(x) = \frac{1}{(x+1)^3}.$$

1soln. ans: $P_4(x) = 1 - 3x + 6x^2 - 10x^3 + 15x^4$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \frac{f^{(n)}(0)}{n!}$
0	$(x+1)^{-3}$	1	$\frac{1}{0!} = 1$
1	$-3(x+1)^{-4}$	-3	$-\frac{3}{1!} = -3$
2	$+3\cdot4(x+1)^{-5}$	$+3\cdot4$	$\frac{3\cdot4}{2!} = \frac{3\cdot4}{2} = 6$
3	$-3\cdot4\cdot5\,(x+1)^{-6}$	$-3\cdot4\cdot5$	$-\frac{3\cdot4\cdot5}{3!} = -\frac{3\cdot4\cdot5}{1\cdot2\cdot3} = -10$
4	$+3\cdot4\cdot5\cdot6\,(x+1)^{-7}$	$+3\cdot4\cdot5\cdot6$	$\frac{3\cdot4\cdot5\cdot6}{4!} = \frac{3\cdot4\cdot5\cdot6}{1\cdot2\cdot3\cdot4} = 15$

Note: Use the term Maclaurin so we know the center is $x_0 = 0$.

2. Find the 10^{th}-order Taylor polynomial $y = P_{10}(x)$, centered about $x_0 = 17$, of the function

$$f(x) = 5 + 6x^7.$$

2soln. ans: $P_{10}(x) = 5 + 6x^7$

Note $f^{(n)}(x) = 0$ for each $n \geq 8$ and $x \in \mathbb{R}$. Let's following notation from Probelm **0B**. If $N \geq 7$, then $f^{(N+1)}(c) = 0$ for any $c \in \mathbb{R}$ and so

$$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!}(x-17)^{N+1} \right| = \frac{0}{(N+1)!}|x-17|^{N+1} = 0\ ,$$

and so $P_N(x) = f(x)$. So $p_{10}(x) = f(x)$.

3. Find a power series representation for

$$f(x) = \ln(10 - x).$$

Hint: $\ln(ab) = \ln a + \ln b$.

3soln. ans: $\ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{n\,10^n}$

$$\ln(10-x) = \ln\left(10\left(1-\frac{x}{10}\right)\right) = \ln 10 + \ln\left[1+\left(\frac{-x}{10}\right)\right]$$ ← used a "Commonly used Taylor Series"

$$= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{-x}{10}\right)^n = \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n \frac{x^n}{10^n}$$

$$= \ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{n\,10^n}$$

$$(-1)^{n+1}(-1)^n = (-1)^{2n+1} = (-1)^{2n}(-1)^1 = -1.$$

I forgot to ask when is this expansion valid so let's do:

Valid $\iff -1 < -\frac{x}{10} \le 1 \iff -10 < x < 10$

4. Consider the function $f(x) = e^x$ over the interval $(-1, 3)$. The 4^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = \sum_{n=0}^{4} \frac{x^n}{n!}.$$

The 4^{th} order Remainder term $R_4(x)$ is defined by $R_4(x) = f(x) - P_4(x)$ and so $e^x \approx P_4(x)$ where the approximation is within an error of $|R_4(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_4(x)|$ that is valid for each $x \in (-1, 3)$.

4soln. ans: $\frac{(e^3)(3^5)}{5!}$

For each $x \in (-1,3)$, the exists c between x & x_0 (so c is also in $(-1,3)$) so that:

$$|R_4(x)| \overset{\text{big theorm}}{=} \left|\frac{f^{(5)}(c)}{5!}x^5\right| = \frac{e^c|x|^5}{5!} \le \frac{e^3 \cdot 3^5}{5!}$$

5. Suppose that the interval of convergence of the series $\sum_{n=1}^{\infty} c_n (x - x_0)^n$ is $(x_0 - R, x_0 + R]$. What can be said about the series at $x = x_0 + R$?

5soln. ans: It must be conditionally convergent.

63. Prove: If the interval of convergence of the series $\sum_{k=0}^{\infty} c_k (x - x_0)^k$ is $(x_0 - R, x_0 + R]$, then the series converges conditionally at $x_0 + R$.

63. The assumption is that $\sum_{k=0}^{\infty} c_k R^k$ is convergent and $\sum_{k=0}^{\infty} c_k (-R)^k$ is divergent. Suppose that $\sum_{k=0}^{\infty} c_k R^k$ is absolutely convergent then $\sum_{k=0}^{\infty} c_k (-R)^k$ is also absolutely convergent and hence convergent because $|c_k R^k| = |c_k (-R)^k|$, which contradicts the assumption that $\sum_{k=0}^{\infty} c_k (-R)^k$ is divergent so $\sum_{k=0}^{\infty} c_k R^k$ must be conditionally convergent.

6. Suppose that the power series $\sum_{n=1}^{\infty} c_n (x - 10)^n$ has interval of convergence $(1, 19)$. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} c_n x^{2n}$?

6soln. ans: $(-3, 3)$

$\Sigma c_n x^{2n}$ converges $\Leftrightarrow$ $\Sigma c_n (x^2)^n$ conv. $\Leftrightarrow$ $\Sigma c_n [(x^2+10)-10]^n$ conv.

$\overset{\text{given}}{\Longleftrightarrow}$ $+1 < x^2 + 10 < 19$ $\Leftrightarrow$ $-9 < x^2 < 9$ $\Leftrightarrow$ $-3 < x < 3$.

7. Describe the motion of a puffo whose position (x, y) is parameterized by

$$x = 6 \sin t$$
$$y = 3 \cos t$$

for $0 \le t \le 2\pi$.

7soln. ans: Moves once clockwise along the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ starting and ending at $(0, 3)$.

Since $\left[\frac{x(t)}{6}\right]^2 + \left[\frac{y(t)}{3}\right]^2 = [\sin t]^2 + [\cos t]^2 = 1$,
the puffo is moving along the ellipse $\left[\frac{x}{6}\right]^2 + \left[\frac{y}{3}\right]^2 = 1$, i.e., along the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$.
He starts at $(x(0), y(0)) = (6 \sin 0, 3 \cos 0) = (0, 3)$.
He finishes at $(x(2\pi), y(2\pi)) = (6 \sin 2\pi, 3 \cos 2\pi) = (0, 3)$.
As he moves from $t = 0$ to $t = 2\pi$, he traces out the ellipse one time.
To figure out if he is moving CW or CCW, note $\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = \left(6 \sin \frac{\pi}{2}, 3 \cos \frac{\pi}{2}\right) = (6, 0)$.
So Mr. Puffo is moving clockwise.

8. Eliminate the parameter to find a Cartesian equation of the curve

$$x = 5e^t$$
$$y = 21e^{-t}.$$

8soln. ans: $y = \dfrac{105}{x}$

$y(t) = 21e^{-t} = \frac{21}{e^t} = \frac{(5)(21)}{5e^t} = \frac{105}{x(t)}$.

9. Find the arc length of the curve

$$x = 3t^2$$
$$y = 2t^3$$

for $0 \le t \le 3$

9soln. ans: $20\sqrt{10} - 2$

$$\begin{aligned}
\text{AL} &= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t=0}^{t=3} \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_{t=0}^{t=3} \sqrt{6^2t^2 + 6^2t^4}\, dt \\
&= \int_{t=0}^{t=3} \sqrt{6^2t^2\left(1+t^2\right)}\, dt = \int_{t=0}^{t=3} 6t\sqrt{1+t^2}\, dt = 3\int_{t=0}^{t=3} (2t)\left(1+t^2\right)^{1/2} dt \\
&\overset{\text{let } u=1+t^2}{\underset{\text{so } du=2t\,dt}{=}} 3\int_{u=1}^{u=10} u^{1/2}\, du = (3)\left(\frac{2}{3}\right) u^{3/2}\Big|_{u=1}^{u=10} = 2u\sqrt{u}\,\Big|_{u=1}^{u=10} = 20\sqrt{10} - 2.
\end{aligned}$$

10. Find a parameterization for the line segment from $(-1, 2)$ to $(10, -6)$ for $0 \le t \le 1$.

10soln. ans: $x = -1 + 11t$ and $y = 2 - 8t$

$$\begin{aligned}
x(t) &= -1 + \left(10 - \left({}^{-}1\right)\right) t = -1 + 11t \\
y(t) &= 2 + \left({}^{-}6 - 2\right) t \qquad = 2 - 8t.
\end{aligned}$$

11. Find an equation of the tangent line to the curve at the point corresponding to $t = 11\pi$.

$$x = t\sin t$$
$$y = t\cos t.$$

11soln. ans: $y = \frac{x}{11\pi} - 11\pi$

$(x(11\pi), y(11\pi)) = (0, -11\pi)$

$$\left.\frac{dy}{dx}\right|_{t=11\pi} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=11\pi} = \left.\frac{\cos t - t\sin t}{\sin t + t\cos t}\right|_{t=11\pi} = \frac{-1-0}{0-11\pi} = \frac{1}{11\pi}.$$

So equation of tangent line to curve when $t = 11\pi$ is

$$\begin{aligned}
\left(y - {}^{-}11\pi\right) &= \frac{1}{11\pi}(x - 0) \\
y + 11\pi &= \frac{1}{11\pi}x \\
y &= \frac{1}{11\pi}x - 11\pi.
\end{aligned}$$