MARK BOX		
PROBLEM	POINTS	
0	10	
1	10	
2	10	
3-10	40=8x5	
11	10	
12	10	
13	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  1. HAND-IN PART. Hand-in only this part.
  2. NOT TO HAND-IN PART. This part will not be collected. Take this part home to learn from and to check your answers when the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For problems > 10**, to receive credit you **MUST**:
  1. **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  2. if a line/box is provided, then:
     — show you work BELOW the line/box
     — put your answer on/in the line/box
  3. if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems (check that you have them all) along with their points.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13th ed., ET): §10.1–10.6 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

0. Fill-in-the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0.1. Geometric Series. Fill in the boxes with the proper range of $r \in \mathbb{R}$.

- The series $\sum r^n$ converges if and only if r satisfies $\boxed{|r| < 1}$.

0.2. p-series. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if $\boxed{p > 1}$.

0.3. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then $\boxed{[\sum b_n \text{ converges} \iff \sum a_n \text{ converges}]}$
- If $L = 0$, then $\boxed{[\sum b_n \text{ converges} \implies \sum a_n \text{ converges}]}$.
- If $L = \infty$, then $\boxed{[\sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}]}$.

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.4. Helpful Intuition Fill in the 3 boxes using: e^x, $\ln x$, x^q. Use each once, and only once.

Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \geq N_q$ then

$$\boxed{\ln x} \leq \boxed{x^q} \leq \boxed{e^x} .$$

0.5. Fill in these boxes with convergent or divergent.

By definition, series $\sum a_n$ is conditionally convergent if and only if

$\sum a_n$ is $\boxed{\text{convergent}}$ and $\sum |a_n|$ is $\boxed{\text{divergent}}$.

1. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true.

Scoring: 2 pts for correct answer, 0 pts for an incorrect answer, 1 pt for a blank answer (indicated by a circled B).

T	[✓ F]	B	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
[✓ T]	F	B	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
[✓ T]	F	B	If $\sum \lvert a_n \rvert$ converges, then $\sum a_n$ converges.
T	[✓ F]	B	If $\sum \lvert a_n \rvert$ diverges, then $\sum a_n$ diverges.
[✓ T]	F	B	If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.

2. Circle the behavior of the given series. Circle up to **1** answers for each problem.
All $\sum$ are undersood as $\sum_{n=2}^{\infty}$

Series	Absolutely Convergent	Conditionally Convergent	Divergent	None of the Others
$\sum \frac{1}{n^2}$	[✓ AC]	CC	DVG	None
$\sum \frac{(-1)^n}{n^2}$	[✓ AC]	CC	DVG	None
$\sum \frac{1}{n}$	AC	CC	[✓ DVG]	None
$\sum \frac{(-1)^n}{n}$	AC	[✓ CC]	DVG	None
$\sum \frac{1}{\sqrt{n}}$	AC	CC	[✓ DVG]	None
$\sum \frac{(-1)^n}{\sqrt{n}}$	AC	[✓ CC]	DVG	None
$\sum \frac{1}{\ln(n)}$	AC	CC	[✓ DVG]	None
$\sum \frac{(-1)^n}{\ln(n)}$	AC	[✓ CC]	DVG	None
$\sum \frac{1}{e^n}$	[✓ AC]	CC	DVG	None
$\sum \frac{(-1)^n}{e^n}$	[✓ AC]	CC	DVG	None

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ For a problem with nothing marked (i.e., left blank) 1 point.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
3	[✓ 3a]	3b	3c	3d	3e					
4	4a	4b	4c	[✓ 4d]	4e					
5	5a	5b	[✓ 5c]	5d	5e					
6	[✓ 6a]	6b	6c	6d	6e					
7	7a	7b	[✓ 7c]	7d	7e					
8	8a	8b	8c	[✓ 8d]	8e					
9	9a	9b	[✓ 9c]	9d	9e					
10	10a	10b	10c	10d	[✓ 10e]					

5	2	1	0
Extra Credit:			

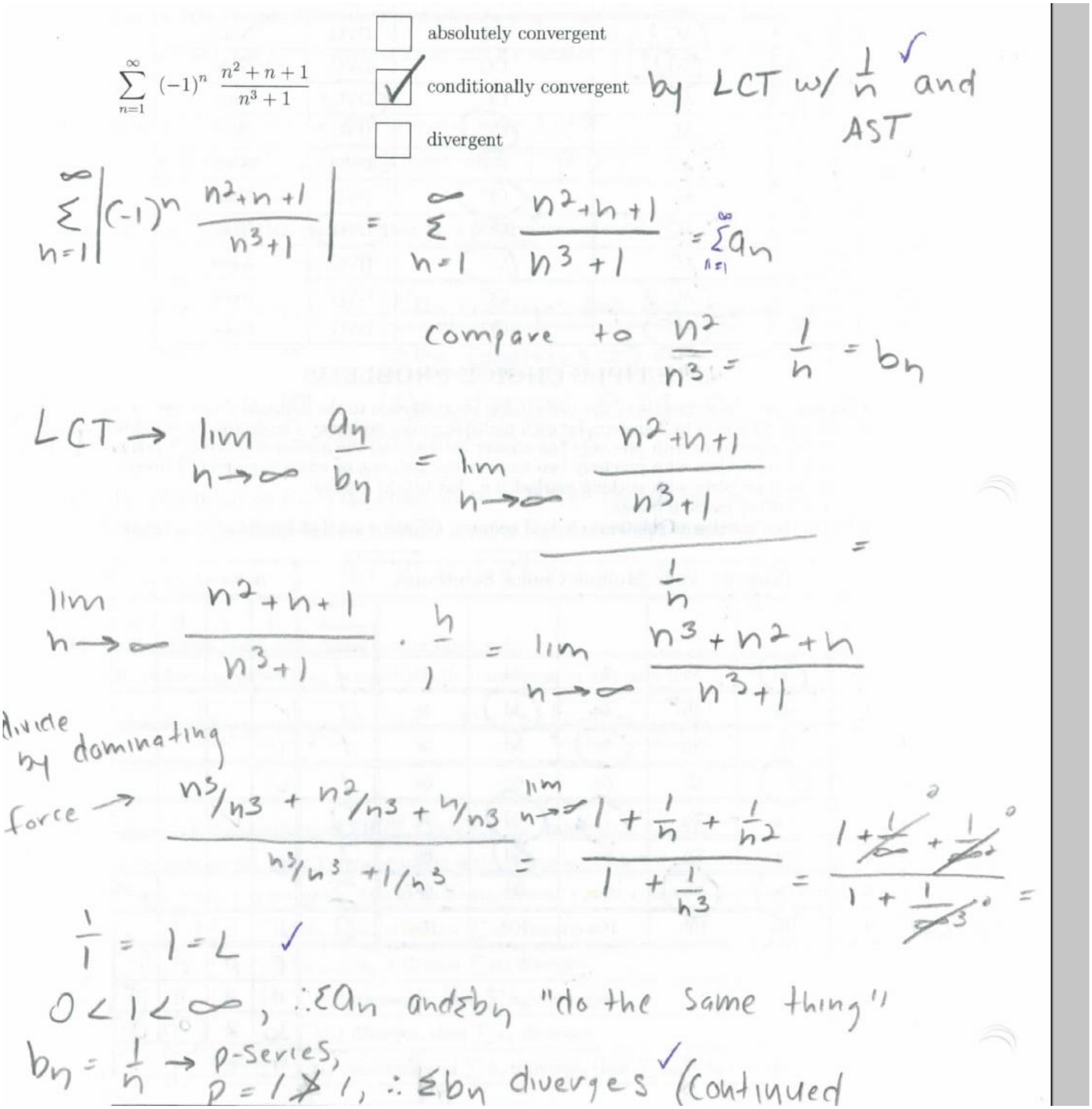
11. Carefully justify the behavior of the series below the choice-boxes and then check the correct choice-box. Be sure to specify which test(s) you are using.

$$\sum_{n=1}^{\infty} (-1)^n \; \frac{n^2+n+1}{n^3+1}$$

☐ absolutely convergent

☐ conditionally convergent

☐ divergent



$$\sum_{n=1}^{\infty} (-1)^n \; \frac{n^2+n+1}{n^3+1}$$

☐ absolutely convergent

☑ conditionally convergent by LCT w/ $\frac{1}{n}$ ✓ and AST

☐ divergent

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^2+n+1}{n^3+1}\right| = \sum_{n=1}^{\infty}\frac{n^2+n+1}{n^3+1} = \sum_{n=1}^{\infty} a_n$$

compare to $\frac{n^2}{n^3} = \frac{1}{n} = b_n$

LCT → $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n^2+n+1}{n^3+1}}{\frac{1}{n}} =$

$\lim_{n\to\infty}\frac{n^2+n+1}{n^3+1}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{n^3+n^2+n}{n^3+1}$

divide by dominating force → $\frac{n^3/n^3 + n^2/n^3 + n/n^3}{n^3/n^3 + 1/n^3} = \lim_{n\to\infty}\frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^3}} = \frac{1+\cancel{\frac{1}{\infty}}^{0}+\cancel{\frac{1}{\infty}}^{0}}{1+\cancel{\frac{1}{\infty^3}}^{0}} =$

$\frac{1}{1} = 1 = L$ ✓

$0 < 1 < \infty$, ∴ Σa_n and Σb_n "do the same thing"

$b_n = \frac{1}{n}$ → p-series, $p = 1 \not> 1$, ∴ Σb_n diverges ✓ (continued

11) (continued)

$|\Sigma|a_n|)$ div g. by LCT

$$u_n = \frac{n^2+n+1}{n^3+1}$$

AST → ☑ pos ✓

☑ $\lim_{n\to\infty} u_n = 0$

☑ decreasing → do 1st derivative test

$$\frac{d}{dn}\ \frac{n^2+n+1 = f}{n^3+1 = g}$$

Quotient rule $\left(\frac{f'g - g'f}{g^2}\right)$

$$\frac{d}{dn} = \frac{(2n+1)(n^3+1) - (3n^2)(n^2+n+1)}{(n^3+1)^2} =$$

$$\frac{\cancel{2n^4}+\cancel{2n}+\cancel{n^3}+1 - (\cancel{3n^4}+\cancel{3n^3}+\cancel{3n^2})}{(n^3+1)^2} =$$

Distribute But still negative

$$\frac{-n^4 \overset{-}{+}4n^3 \overset{-}{+}3n^2+2n+1}{(n^3+1)^2} \to \frac{-}{+},$$

$\frac{d}{dn}u_n < 0$ ∴ AST holds true,

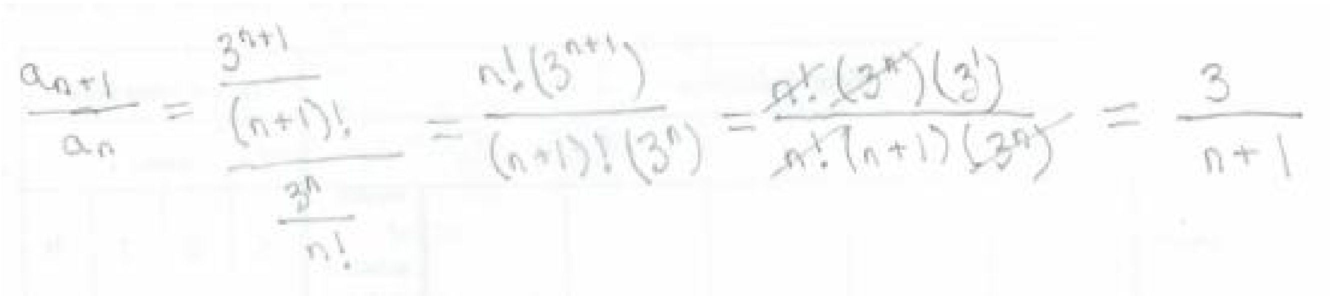
Σa_n is convergent ✓

12. Let

$$a_n = \frac{3^n}{n!}$$

12.1. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{3}{n+1}}$$



$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{n!(3^{n+1})}{(n+1)!(3^n)} = \frac{\cancel{n!}(\cancel{3^n})(3^1)}{\cancel{n!}(n+1)(\cancel{3^n})} = \frac{3}{n+1}$$

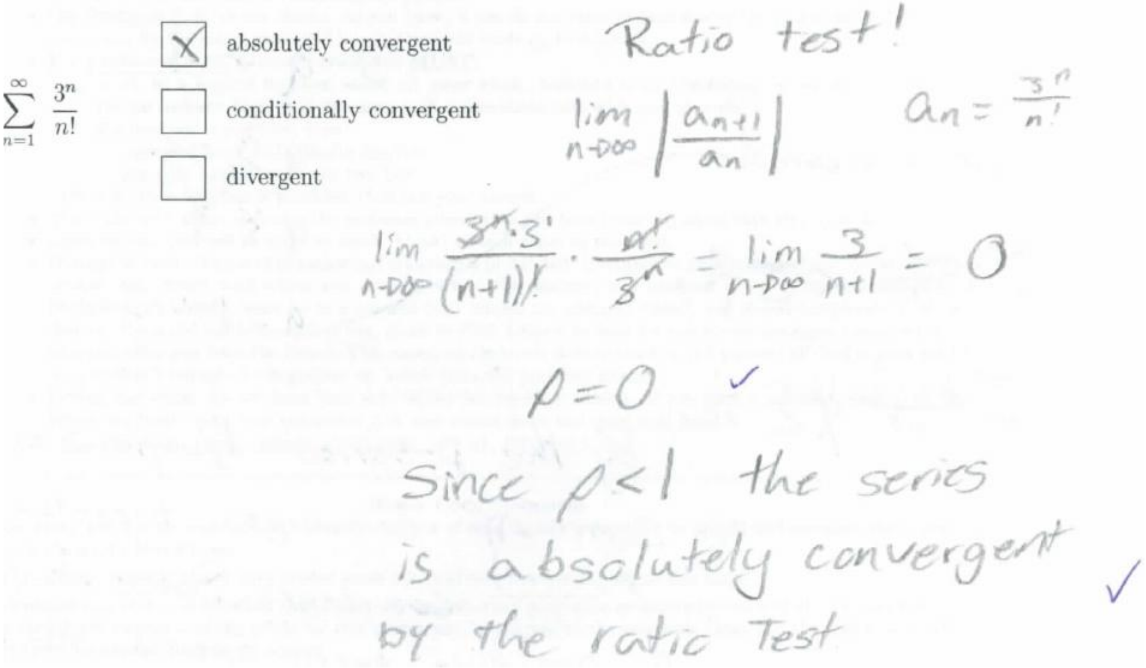
12.2. Carefully justify the behavior of the series below the choice-boxes and then check the correct choice-box. Be sure to specify which test(s) you are using. You may use part 12.1.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

☐ absolutely convergent

☐ conditionally convergent

☐ divergent



$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

☒ absolutely convergent

☐ conditionally convergent

☐ divergent

Ratio test!

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \qquad a_n = \frac{3^n}{n!}$$

$$\lim_{n\to\infty} \frac{3^n \cdot 3^1}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

$\rho = 0$ ✓

Since $\rho < 1$ the series is absolutely convergent by the ratio Test ✓

13. Carefully justify the behavior of the series below the choice-boxes and then check the correct choice-box. Be sure to specify which test(s) you are using.

$$\sum_{n=1}^{\infty} \frac{\ln(n!)}{n^3}$$

☒ absolutely convergent

☐ ~~conditionally convergent~~ b/c positive-termed series

☐ divergent

This problem is number 13 from our *15 Serious Series' Problems.* First note that

$$\frac{\ln(n!)}{n^3} \geq 0 \tag{13.1}$$

and so $\sum \frac{\ln(n!)}{n^3}$, as a postive-termed series, is either absolutely convergent or divergent. I.e., $\sum \frac{\ln(n!)}{n^3}$ cannot be conditionally convergent.

Next note that

$$\ln(n!) = \ln(1 \cdot 2 \cdots n) = \ln 1 + \ln 2 + \ldots \ln n \ \leq n \ \ln n \ . \tag{13.2}$$

Way 1:

So for n sufficiently large (i.e., big enough)

$$0 \overset{\text{by (13.1)}}{\leq} \frac{\ln(n!)}{n^3} \overset{\text{by (13.2)}}{\leq} \frac{n \ \ln n}{n^3} \overset{\textcircled{A}}{=} \frac{\ln n}{n^2} \underset{\text{by Helfpul Intuition}}{\overset{\text{for } n \text{ big}}{\leq}} \frac{n^{\frac{1}{2}}}{n^2} \overset{\textcircled{A}}{=} \frac{1}{n^{\frac{3}{2}}} \tag{13.3}$$

The series $\sum_{n=1}^{\infty} \dfrac{1}{n^{\frac{3}{2}}}$ is a p-series, with $p = \frac{3}{2} > 1$ so $\sum \dfrac{1}{n^{\frac{3}{2}}}$ converges.

By the DCT, using $c_n = \frac{1}{n^{\frac{3}{2}}}$ for comparison, $\sum \frac{\ln(n!)}{n^3}$ converges.

Way 2:

For any

$$q > 0 \tag{13.4}$$

fo for n sufficiently large (i.e., big enough)

$$0 \overset{\text{by (13.1)}}{\leq} \frac{\ln(n!)}{n^3} \overset{\text{by (13.2)}}{\leq} \frac{n \ \ln n}{n^3} \overset{\textcircled{A}}{=} \frac{\ln n}{n^2} \underset{\text{by Helfpul Intuition}}{\overset{\text{for } n \text{ big}}{\leq}} \frac{n^q}{n^2} \overset{\textcircled{A}}{=} \frac{1}{n^{2-q}} \tag{13.5}$$

(Thinking land: we have just bounded above so going for convergence.)

The series $\sum \dfrac{1}{n^{2-q}}$ is a p-series, with $p = 2 - q$, so if

$$2 - q > 1 \tag{13.6}$$

then $\sum \dfrac{1}{n^{2-q}}$ converges. So we want, by (13.4) and (13.6)

$$0 < q \qquad \text{and} \qquad 1 < 2 - q$$

Since $[1 < 2 - q] \Leftrightarrow [q < 2 - 1]$, we want

$$0 < q < 1.$$

Now now just pick any $0 < q < 1$ (in Way 1, we take $q = \frac{1}{2}$) and proceed as in Way 1.

Common Algebra Mistake:

Many students first tried the Ratio Test first, which is a good first try given the factorial. So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\ln((n+1)!)}{(n+1)^3} \ \frac{n^3}{\ln((n!))} \overset{\textcircled{A}}{=} \left[\frac{\ln((n+1)!)}{\ln((n!))}\right] \ \left[\left(\frac{n}{n+1}\right)^3\right].$$

Since $\frac{n}{n+1} \xrightarrow{n\to\infty} 1$, we know $\left(\frac{n}{n+1}\right)^3 \xrightarrow{n\to\infty} 1^3 = 1$. So we just have to figure out $\lim_{n\to\infty} \frac{\ln\left((n+1)!\right)}{\ln((n!))}$, which is where the COMMON ALGEBRA MISTAKE occurred. We now that

$$\ln\left(\frac{b}{a}\right) = \ln b - \ln a \qquad\qquad \text{but} \qquad\qquad \frac{\ln b}{\ln a} \neq \ln\left(\frac{b}{a}\right).$$

(if you have a hard time remebering this, think about what happens when $a = 1$, for then $\ln a = \ln 1 = 0$) so

$$\frac{\ln\left((n+1)!\right)}{\ln\left((n!)\right)} \neq \ln\left(\frac{(n+1!)}{n!}\right)$$

So

$$\left|\frac{a_{n+1}}{a_n}\right| = \left[\frac{\ln\left((n+1)!\right)}{\ln\left((n!)\right)}\right] \left[\left(\frac{n}{n+1}\right)^3\right] \neq \ln\left(\frac{(n+1!)}{n!}\right) \left[\left(\frac{n}{n+1}\right)^3\right].$$

Some students who made this mistake also had troubles with their next step:

$$\ln\left(\frac{(n+1)!}{n!}\right) \overset{\text{Ⓐ}}{=} \ln\left(n+1\right) \xrightarrow{n\to\infty} \infty\ .$$

NOT TO HAND-IN PART
STATEMENT OF MULTIPLE CHOICE PROBLEMS

3. Evaluate

$$\lim_{n\to\infty} \frac{{}^-9n^2 - 8n + 7}{7n - 6}\,.$$

3soln. Divide numerator and denominator by the *dominating force*, which is n^2. So

$$\frac{{}^-9n^2 - 8n + 7}{7n - 6} = \frac{\frac{{}^-9n^2-8n+7}{n^2}}{\frac{7n-6}{n^2}} = \frac{{}^-9 - \frac{8}{n} + \frac{7}{n^2}}{\frac{7}{n} - \frac{6}{n^2}} \xrightarrow{n\to\infty} {}^-\infty$$

4. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt[2]{9n^2 - 8n + 7}}{\sqrt[3]{8n^3 + 7n^2 - 6n - 5}}\,.$$

4soln. The numerator's *dominating force* is $\sqrt[2]{n^2} = n$ while the denominator's *dominating force* is $\sqrt[3]{n^3} = n$ so the fraction's *dominating force* is n. So divide through by n.

$$\frac{\sqrt[2]{9n^2-8n+7}}{\sqrt[3]{8n^3+7n^2-6n-5}} = \frac{\frac{\sqrt[2]{9n^2-8n+7}}{n}}{\frac{\sqrt[3]{8n^3+7n^2-6n-5}}{n}} = \frac{\sqrt[2]{\frac{9n^2-8n+7}{n^2}}}{\sqrt[3]{\frac{8n^3+7n^2-6n-5}{n^3}}} = \frac{\sqrt[2]{9 - \frac{8}{n} + \frac{7}{n^2}}}{\sqrt[3]{8 + \frac{7}{n} - \frac{6}{n^2} - \frac{5}{n^3}}} \xrightarrow{n\to\infty} \frac{\sqrt[2]{9 - 0 + 0}}{\sqrt[3]{8 + 0 - 0 - 0}} = \frac{3}{2}.$$

5. Evaluate

$$\lim_{n\to\infty} \left(\frac{n}{n+17}\right)^{2n}\,.$$

5soln. This problem was from your Maple Assignment 3. There are two ways to do this problem. We can either evaluate the limit directly or we can modify the function algebraically to make our life (a little) easier. This comes from the observation that

$$\frac{x}{x+17} = \frac{x + 17 - 17}{x+17} = \frac{x+17}{x+17} + \frac{-17}{x+17} = 1 + \frac{-17}{x+17}$$

Thus

$$\lim_{x\to\infty} \left(\frac{x}{x+17}\right)^{2x} = \lim_{x\to\infty} \left(1 + \frac{-17}{x+17}\right)^{2x}$$

and the limit on the right can be evaluated similarly to a limit we have seen in recitation. We will compute both limits in these solutions.

To find the limit on the left (the original function without any algebraic manipulations), observe that since $\lim_{x\to\infty} \left(\frac{x}{x+17}\right)^{2x}$ is of the indeterminate form 1^∞, as the L'Hopital handout suggests, we want to first consider $\ln\left(\left(\frac{x}{x+17}\right)^{2x}\right)$ since this will allow us to *bring the exponent down* and then rewrite using algebraic manipulations so as to be able to apply L'H's rule. Whenever we do this, we just have to remember to exponentiate our result at the end.

$$\lim_{x\to\infty} \ln\left(\left(\frac{x}{x+17}\right)^{2x}\right) \overset{\textcircled{A}}{=} \lim_{x\to\infty} 2x\,\ln\left(\frac{x}{x+17}\right) \underset{\textcircled{A}}{\overset{\cancel{\infty\cdot 0}}{=}} 2\lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+17}\right)}{\frac{1}{x}} \underset{\text{L'H}}{\overset{\xcancel{\frac{0}{0}}}{=}} 2\lim_{x\to\infty} \frac{D_x \ln\left(\frac{x}{x+17}\right)}{D_x\left(\frac{1}{x}\right)}$$

$$\overset{\textcircled{c}}{=} 2\frac{\left[\frac{1}{\frac{x}{x+17}}\right] \cdot \left[D_x\left(\frac{x}{x+17}\right)\right]}{D_x\left(\frac{1}{x}\right)} \overset{\textcircled{c}}{=} 2\frac{\left[\frac{1}{\frac{x}{x+17}}\right] \cdot \left[\frac{17}{(x+17)^2}\right]}{-\frac{1}{x^2}} \overset{\textcircled{A}}{=} \lim_{x\to\infty} 2\frac{-17x^2}{x^2+17x}$$

$$= \frac{2(-17)}{1} = -34\,.$$

To find the limit on the right, (the one we algebraically modified), do the exact same thing by considering ln() of the limit. We know we will get -34, since the limits are equal.

$$\lim_{x\to\infty} \ln\left(\left(1+\frac{-17}{x+17}\right)^{2x}\right) \overset{\textcircled{A}}{=} \lim_{x\to\infty} 2x\,\ln\left(1+\frac{-17}{x+17}\right) \underset{\textcircled{A}}{\overset{\cancel{\infty\cdot 0}}{=}} 2\lim_{x\to\infty} \frac{\ln\left(1+\frac{-17}{x+17}\right)}{\frac{1}{x}} \underset{\text{L'H}}{\overset{\xcancel{\frac{0}{0}}}{=}} 2\lim_{x\to\infty} \frac{D_x \ln\left(1+\frac{-17}{x+17}\right)}{D_x\left(\frac{1}{x}\right)}$$

$$\overset{\textcircled{c}}{=} 2\frac{\left[\frac{1}{1+\frac{-17}{x+17}}\right] \cdot \left[D_x\left(1+\frac{-17}{x+17}\right)\right]}{D_x\left(\frac{1}{x}\right)} \overset{\textcircled{c}}{=} 2\frac{\left[\frac{1}{1+\frac{-17}{x+17}}\right] \cdot \left[(-17)\;D_x\left(\frac{1}{x+17}\right)\right]}{D_x\left(\frac{1}{x}\right)}$$

$$\overset{\textcircled{c}}{=} 2\lim_{x\to\infty} \frac{-17}{1+\frac{-17}{x+17}} \cdot \lim_{x\to\infty} \frac{-x^2}{-(x+17)^2} \overset{\textcircled{A}}{=} 2\lim_{x\to\infty} \frac{-17}{1+\frac{-17}{x+17}} \cdot (1) = \frac{-34}{1+0} = -34\,.$$

We just found that

$$\lim_{x\to\infty} \ln\left(\left(\frac{x}{x+17}\right)^{2x}\right) = -34$$

and so

$$\lim_{x\to\infty} \left(\frac{x}{x+17}\right)^{2x} = \lim_{x\to\infty} e^{\ln\left(\left(\frac{x}{x+17}\right)^{2x}\right)} = e^{\lim_{x\to\infty} \ln\left(\left(\frac{x}{x+17}\right)^{2x}\right)} = e^{-34}\,.$$

So

$$\lim_{n\to\infty} \left(\frac{n}{n+17}\right)^{2n} = \boxed{e^{-34}}\,.$$

6. Consider the formal series

$$\sum_{n=2}^{\infty} \left(\frac{2n+3}{3n+2}\right)^{n}\,.$$

6soln. Answer: The series converges by the Root Test.

$$\left|\left(\frac{2n+3}{3n+2}\right)^{n}\right|^{\frac{1}{n}} = \frac{2n+3}{3n+2} = \frac{2}{3} := \rho < 1.$$

7. Find **<u>all</u>** real numbers r satisfying that

$$\sum_{n=2}^{\infty} r^n = \frac{1}{6}\,.$$

7soln. Soln: $\dfrac{-1}{2}$ and $\dfrac{1}{3}$

<u>First</u> note that for the series $\sum_{n=2}^{\infty} r^n$ to converge (so that the problem even makes sense), we need

$$|r| < 1.$$

So let $\boxed{|r| < 1}$. Next, to find the sum $\sum_{n=2}^{\infty} r^n$, consider the partial sums $s_n \overset{\text{def}}{=} r^2 + r^3 + \ldots + r^{n-1} + r^n$.

Cancellation Heaven occurs with a geometric series when one computes $s_n - rs_n$. Let's see why.

$$\begin{array}{lcl} s_n & = & r^2 + r^3 + \ldots + r^{n-1} + r^n \\ r\,s_n & = & r^3 + r^4 + \ldots + r^n + r^{n+1} \end{array}$$

Do you see the cancellation that would occur if we take $s_n - rs_n$?

$$\begin{array}{lcccccccccc} s_n & = & r^2 & + & \cancel{r^3} & + & \ldots & + & \cancel{r^{n-1}} & + & \cancel{r^n} \\ & & & \swarrow & & & \cdot\swarrow\cdot & & & \swarrow & \\ r\,s_n & = & \cancel{r^3} & + & \cancel{r^4} & + & \ldots & + & \cancel{r^n} & + & r^{n+1} \end{array}$$

substract ―――――――――――――――――――――――――――――――――――

$$(1-r)\,s_n \overset{\textcircled{A}}{=} s_n - rs_n = r^2 \qquad\qquad\qquad - r^{n+1}$$

and since $r \neq 1$, then

$$s_n = \frac{r^2 - r^{n+1}}{1-r} \xrightarrow{\text{since } |r|<1} \frac{r^2}{1-r} = \sum_{n=2}^{\infty} r^n.$$

So we are looking for $r \in \mathbb{R}$ so that $|r| < 1$ and $\frac{r^2}{1-r} = \frac{1}{6}$. Note $\left[\frac{r^2}{1-r} = \frac{1}{6}\right] \Leftrightarrow [6r^2 = 1 - r] \Leftrightarrow [6r^2 + r - 1 = 0] \Leftrightarrow$

$$r = \frac{-1 \pm \sqrt{1 + 4\,(6)}}{2\,(6)} = \frac{-1 \pm 5}{12} = \begin{cases} \frac{-1+5}{12} = \frac{4}{12} = \frac{1}{3} \\ \frac{-1-5}{12} = -\frac{6}{12} = -\frac{1}{2}. \end{cases}$$

8. The series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.23}}$$

is

8soln. Answer: absolutely convergent by the Direct Comparison Test, using for comparison $\frac{1}{n^{1.01}}$

* useful entuition!

$\ln n \le n^q \le e^n$; for $q>0$ and some value of n large enough.

$\frac{\ln n}{n^{1.23}} \le \frac{n^q}{n^{1.23}}$; $\frac{n^q}{n^{1.23}} = \frac{1}{n^{1.23-q}}$

USE DIRECT COMPARISON TEST

- $\sum a_n$ is bounded above by $\sum \frac{1}{n^{1.23-q}}$;
- to converge $\sum \frac{1}{n^{1.23-q}}$, a p-series, $1.23-q=p>1$

⊕ $1.23-q>1 \Rightarrow -q>-0.23 \Rightarrow q<0.23$

- $q=0.22$

Since $\frac{\ln n}{n^{1.23}} \le \frac{1}{n^{1.01}}$

and $\sum \frac{1}{n^{1.01}}$ is a p-series w/ $p>1$

it follows that $\sum \frac{\ln n}{n^{1.23}}$ also converges.

nice!

9. Let c be a natural number (i.e., $c \in \{1, 2, 3, 4, \ldots\}$). The series

$$\sum_{n=1}^{\infty} \frac{(n!)^6}{(cn)!}$$

9soln. Soln: diverges when $c < 6$ and converges when $c \geq 6$

Let c be a natural number (i.e., $c \in \{1, 2, 3, 4, \ldots\}$).

The series $\sum_{n=1}^{\infty} \frac{(n!)^6}{(cn)!} \frac{(n!)^6}{(cn)!}$ diverges when $c < 6$ and converges when $c \geq 6$.

Let $a_n = \frac{(n!)^6}{(cn)!}$ and $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{[(n+1)!]^6}{[n!]^6} \frac{(cn)!}{(cn+c)!} = \frac{(n+1)^6}{(cn+1)(cn+2)\cdots(cn+c)}$$
$$= \frac{n^6 + (\text{a poly. of degree atmost 5})}{c^c\,(n^c) + (\text{a poly. of degree at most (c-1)})} \ .$$

If $c < 6$, then $\rho = \infty$. If $c > 6$, then $\rho = 0$, If $c = 6$, then $\rho = \frac{1}{6^6} < 1$. Now apply the ratio test.

10. What is the smallest integer N such that the Alternating Series Estimate/Remainder Theorem guarentees that

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{N} \frac{(-1)^n}{n}\right| \leq 0.05?$$

Note that $0.05 = \frac{0.05}{1.0000} = \frac{5}{100} = \frac{1}{20}$.

10soln. Soln: None of the other.

Note that $0 \leq \frac{1}{n} \searrow 0$ so the AST applies and we know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges and that

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{N} \frac{(-1)^n}{n}\right| \leq \frac{1}{N+1}.$$

So

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{N} \frac{(-1)^n}{n}\right| \overset{\text{have}}{\leq} \frac{1}{N+1} \overset{\text{want}}{\leq} \frac{1}{20}.$$

Note

$$\left[\frac{1}{N+1} \leq \frac{1}{20}\right] \Leftrightarrow (20 \leq N+1] \Leftrightarrow (19 \leq N] \, .$$