MARK BOX		
PROBLEM	POINTS	
0	15	
1-11	55=11x5	
12	10	
13	10	
14	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  1. HAND IN PART. Hand in only this part.
  2. STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **MultipleChoice problems 1–11**, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
- **For problems > 11**, to receive credit you **MUST**:
  1. **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  2. if a line/box is provided, then:
     — show you work BELOW the line/box
     — put your answer on/in the line/box
  3. if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1-8.5, 8.7, 8.8 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in-the blanks/boxes.

0.1. If $u \neq 0$, then $\int \frac{du}{u} =$ ___ $\ln\ |u|$ ___ + C

0.2. $\int \tan u\, du \ =$ ___ $\ln|\sec u| \overset{or}{=} - \ln|\cos u|$ ___ $+C$

0.3. $\int \sec u\, du \ =$ ___ $\ln|\sec u + \tan u| \overset{or}{=} - \ln|\sec u - \tan u|$ ___ $+C$

0.4. $\int \sec^2 u\, du \ =$ ___ $\tan u$ ___ $+C$

0.5. $\int \sec u \tan u\, du \ =$ ___ $\sec u$ ___ $+C$

0.6. Integration by parts formula: $\int u\, dv \ =$ ___ $uv - \int v\, du$ ___

0.7. Trig sub.: if the integrand involves $a^2 - u^2$, then one makes the substitution $u \ =$ ___ $a \sin\theta$ ___

0.8. Trig sub.: if the integrand involves $u^2 - a^2$, then one makes the substitution $u \ =$ ___ $a \sec\theta$ ___

0.9. trig formula ... your answer should involve trig functions of θ, and not of 2θ: $\cos(2\theta) =$ ___ $\cos^2\theta - \sin^2\theta$ ___ .

0.10. trig formula ... your answer should involve trig functions of θ, and not of 2θ: $\sin(2\theta) =$ ___ $2\sin\theta\cos\theta$ ___ .

0.11. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\cos^2(\theta) \ = \ \frac{1}{2}$ ___ $(1 + \cos(2\theta))$ ___ .

0.12. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\sin^2(\theta) \ = \ \frac{1}{2}$ ___ $(1 - \cos(2\theta))$ ___ .

0.13. trig formula ... since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween tangent (i.e., tan) and secant (i.e., sec) is ___ $1 + \tan^2\theta = \sec^2\theta$ ___ .

0.14. arctan(-1) = ___ $-\dfrac{\pi}{4}$ ___ **RADIANS**. (your answer should be an angle)

0.15. If $f\colon [a, b) \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_a^b f(x)\ dx$ by

$$\int_a^b f(x)\ dx = \boxed{\lim_{t \to b^-} \int_a^t f(x)\ dx}\ .$$

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ For a problem with nothing marked (i.e., left blank) 1 point.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Table for Your Muliple Choice Solutions — Do Not Write Below

PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	(1c)	1d	1e					
2	2a	2b	(2c)	2d	2e					
3	3a	(3b)	3c	3d	3e					
4	4a	4b	4c	(4d)	4e					
5	5a	5b	(5c)	5d	5e					
6	6a	(6b)	6c	6d	6e					
7	7a	7b	7c	(7d)	7e					
8	(8a)	8b	8c	8d	8e					
9	9a	(9b)	9c	9d	9e					
10	10a	10b	(10c)	10d	10e					
11	11a	11b	11c	(11d)	11e					

5	2	1	0
Extra Credit:			

For problems 12-14, put your answer in the box and show your work below the box.

12. $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx \;=\; \ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C$ also acceptable is $\ln\left|\sqrt{x^2+4}+x\right|$ + C

On this problem, your final answer should not have a trig function in it.

This is number 10 from the 100 Integrals.

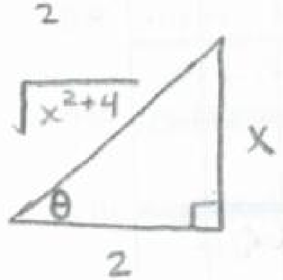
$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$\int\frac{1}{\sqrt{x^2+4}}\,dx = \int\frac{2\sec^2\theta\,d\theta}{\sqrt{(2\tan\theta)^2+4}} = \int\frac{2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta+4}} = \int\frac{2\sec^2\theta\,d\theta}{\sqrt{4(\tan^2\theta+1)}} = \int\frac{2\sec^2\theta\,d\theta}{\sqrt{4\sec^2\theta}}$$

$$\int\frac{2\sec^2\theta\,d\theta}{\sqrt{4\sec^2\theta}} = \int\frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C$$

$\frac{x}{2} = \tan\theta$ SOHCAHTOA



so $\sec\theta = \frac{\sqrt{x^2+4}}{2}$

and $\tan\theta = \frac{x}{2}$

thus $\ln|\sec\theta+\tan\theta| = \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right|$

$x^2+2^2 = c^2$

$\sqrt{x^2+4} = c$

Why are both solutions acceptable? Well, they are both correct since

$$\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+K \;=\; \ln\left|\left(\frac{1}{2}\right)\left(\sqrt{x^2+4}+x\right)\right|+K \;=\; \ln\frac{1}{2}+\ln\left|\sqrt{x^2+4}+x\right|+K$$

$$=\; \ln\left|\sqrt{x^2+4}+x\right|+\left(K+\ln\frac{1}{2}\right)\;.$$

13. $\int e^{3x}\cos 5x\ dx \ = \ \frac{e^{3x}}{34}\ (3\cos 5x + 5\sin 5x)$ + C

Way # 1

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{3x} \qquad dv_1 = \cos 5x\, dx$$
$$du_1 = 3e^{3x}\, dx \qquad v_1 = \frac{1}{5}\sin 5x\ .$$

So by integration by parts

$$\int e^{3x}\ \cos 5x\ dx \ = \ \frac{1}{5}e^{3x}\sin 5x - \frac{3}{5}\int e^{3x}\sin 5x\, dx\ .$$

Now let

$$u_2 = e^{3x} \qquad dv_2 = \sin 5x\, dx$$
$$du_2 = 3e^{3x}\, dx \qquad v_2 = \frac{-1}{5}\cos 5x\ .$$

to get

$$\begin{aligned}\int e^{3x}\ \cos 5x\ dx \ &= \ \frac{1}{5}e^{3x}\sin 5x - \frac{3}{5}\left[\frac{-1}{5}e^{3x}\cos 5x - \frac{-3}{5}\int e^{3x}\cos 5x\, dx\right] \\ &= \ \frac{1}{5}e^{3x}\sin 5x + \frac{3}{5^2}e^{3x}\cos 5x - \frac{3^2}{5^2}\int e^{3x}\cos 5x\, dx\ .\end{aligned}$$

Now solving for $\int e^{3x}\cos 5x\, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{3^2}{5^2}\right]\int e^{3x}\ \cos 5x\ dx \ = \ \frac{1}{5}e^{3x}\sin 5x + \frac{3}{5^2}e^{3x}\cos 5x + K$$

and so

$$\begin{aligned}\int e^{3x}\ \cos 5x\ dx \ &= \ \left[\frac{5^2}{34}\right]\left(\frac{1}{5}e^{3x}\sin 5x + \frac{3}{5^2}e^{3x}\cos 5x + K\right) \\ &= \ \frac{5}{34}\ e^{3x}\sin 5x + \frac{3}{34}\ e^{3x}\cos 5x + \left[\frac{K5^2}{34}\right] \\ &= \ \frac{e^{3x}}{34}\ (5\sin 5x + 3\cos 5x) + \left[\frac{K5^2}{34}\right]\ .\end{aligned}$$

Thus

$$\int e^{3x}\ \cos 5x\ dx \ = \ \boxed{\frac{e^{3x}}{34}\ (3\cos 5x + 5\sin 5x) + C}\ .$$

Way # 2

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$u_1 = \cos 5x \qquad dv_1 = e^{3x}dx$$

$$du_1 = -5\sin 5x\,dx \qquad v_1 = \frac{1}{3}\,e^{3x}\ .$$

So, by integration by parts

$$\int e^{3x}\cos 5x\ dx \ = \ \frac{1}{3}e^{3x}\cos 5x - \frac{-5}{3}\int e^{3x}\sin 5x\,dx\ .$$

Now let

$$u_2 = \sin 5x \qquad dv_2 = e^{3x}dx$$

$$du_2 = 5\cos 5x\,dx \qquad v_2 = \frac{1}{3}\,e^{3x}\ .$$

to get

$$\begin{aligned}\int e^{3x}\cos 5x\ dx \ &= \ \frac{1}{3}e^{3x}\cos 5x + \frac{5}{3}\left[\frac{1}{3}e^{3x}\sin 5x - \frac{5}{3}\int e^{3x}\cos 5x\,dx\right] \\ &= \ \frac{1}{3}e^{3x}\cos 5x + \frac{5}{3^2}e^{3x}\sin 5x - \frac{5^2}{3^2}\int e^{3x}\cos 5x\,dx\ .\end{aligned}$$

Now solving for $\int e^{3x}\cos 5x\,dx$ (use the *bring to the other side* idea) we get

$$\left[1+\frac{5^2}{3^2}\right]\int e^{3x}\cos 5x\ dx \ = \ \frac{1}{3}e^{3x}\cos 5x + \frac{5}{3^2}e^{3x}\sin 5x + K$$

and so

$$\begin{aligned}\int e^{3x}\cos 5x\ dx \ &= \ \left[\frac{3^2}{3^2+5^2}\right]\left(\frac{1}{3}e^{3x}\cos 5x + \frac{5}{3^2}e^{3x}\sin 5x + K\right) \\ &= \ \frac{3}{34}\,e^{3x}\cos 5x + \frac{5}{34}\,e^{3x}\sin 5x + \left[\frac{K3^2}{3^2+5^2}\right] \\ &= \ \frac{e^{3x}}{34}\,(3\cos 5x + 5\sin 5x) + \left[\frac{K3^2}{3^2+5^2}\right]\end{aligned}$$

Thus

$$\int e^{3x}\cos 5x\ dx \ = \ \boxed{\frac{e^{3x}}{34}\,(3\cos 5x \ + \ 5\sin 5x) + C}\ .$$

Doesn't Work Way

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

14. Let n be an integer greater than 2 (so $n \in \{3, 4, 5, 6, \ldots\}$). Find a reduction formula for $\int \sec^n x$.

$$\int \sec^n x \, dx \;=\; \left(\frac{1}{n-1}\right) \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

EXAMPLE 6 Find a reduction formula for $\int \sec^n x \, dx$.

Solution The idea is that n is a (large) positive integer, and that we want to express the given integral in terms of a lower power of $\sec x$. The easiest power of $\sec x$ to integrate is $\sec^2 x$, so we proceed as follows.

We don't choose $dv = \sec x \, dx$ because this would introduce a natural logarithm function, a fearsome complication in the second integration.

Let $\quad u = \sec^{n-2} x, \qquad dv = \sec^2 x \, dx.$

Then $\quad du = (n-2) \sec^{n-2} x \tan x \, dx, \qquad v = \tan x.$

This gives

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$
$$= \sec^{n-2} x \tan x - (n-2) \int (\sec^{n-2} x)(\sec^2 x - 1) \, dx.$$

Hence

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx.$$

We solve this equation for the desired integral and find that

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx. \tag{5}$$

This is the desired reduction formula. For example, if we take $n = 3$ in this formula, we find that

$$\int \sec^3 x \, dx = \tfrac{1}{2} \sec x \tan x + \tfrac{1}{2} \int \sec x \, dx$$
$$= \tfrac{1}{2} \sec x \tan x + \tfrac{1}{2} \ln|\sec x + \tan x| + C. \tag{6}$$

In the last step we used Equation (15) of Section 8.2,

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

The reason for using the reduction formula in (5) is that repeated application must yield one of the two elementary integrals $\int \sec x \, dx$ and $\int \sec^2 x \, dx$. For instance, with $n = 4$ we get

$$\int \sec^4 x \, dx = \tfrac{1}{3} \sec^2 x \tan x + \tfrac{2}{3} \int \sec^2 x \, dx$$
$$= \tfrac{1}{3} \sec^2 x \tan x + \tfrac{2}{3} \tan x + C, \tag{7}$$

and with $n = 5$ we get

$$\int \sec^5 x \, dx = \tfrac{1}{4} \sec^3 x \tan x + \tfrac{3}{4} \int \sec^3 x \, dx$$
$$= \tfrac{1}{4} \sec^3 x \tan x + \tfrac{3}{8} \sec x \tan x + \tfrac{3}{8} \ln|\sec x + \tan x| + C, \tag{8}$$

using in the last step the formula in Equation (6).

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.

---

1. Evaluate

$$\int_{x=0}^{x=\frac{\pi}{2}} \sin x\, dx.$$

1soln. $\int_{x=0}^{x=\frac{\pi}{2}} \sin x\, dx \;=\; -\cos x \,\Big|_{x=1}^{x=\pi/2} \;=\; \left(-\cos\frac{\pi}{2}\right) - (-\cos 0) \;=\; (0) - (-1) = 1$

2. Evaluate

$$\int_{x=0}^{x=\sqrt{\frac{\pi}{2}}} x \sin\left(x^2\right)\, dx$$

2soln. First do indefinite integral. Simple u-du substitution with $u = x^2$ and so $du = 2xdx$.

$$\int x \sin\left(x^2\right)\, dx = \frac{1}{2}\int \sin\left(x^2\right) \boxed{2x\, dx} = \frac{1}{2}\int \sin u\, du = \frac{-\cos u}{2} + C = \frac{-\cos x^2}{2} + C\ .$$

Next check indefinite integral:

$$D_x\left(\frac{-\cos x^2}{2}\right) = \frac{-1}{2} D_X\left(\cos\left(x^2\right)\right) = \frac{-1}{2}\left({}^{-}\sin x^2\right)(2x) = x \sin x^2 \quad \boxed{\checkmark}\ .$$

So

$$\int_{x=0}^{x=\sqrt{\frac{\pi}{2}}} x \sin\left(x^2\right)\, dx = \frac{-\cos\left(x^2\right)}{2}\Big|_{x=0}^{x=\sqrt{\frac{\pi}{2}}} = \left(\frac{-\cos\frac{\pi}{2}}{2}\right) - \left(\frac{-\cos 0}{2}\right) = (0) - \left(\frac{-1}{2}\right) = \frac{1}{2}\ .$$

3. Evaluate

$$\int_{x=0}^{x=\frac{\pi}{4}} \sin^2 x\, dx$$

3soln. First do indefinite integral.

Solution Here we make use of half-angle identities.

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int dx - \frac{1}{4}\int (\cos 2x)(2\,dx)\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

Next check indefinite integral:

$$D_x\left(\frac{x}{2}-\frac{\sin 2x}{4}\right) = \frac{1}{2} - \frac{D_x\left(\sin 2x\right)}{4} = \frac{1}{2} - \frac{\left(2\cos 2x\right)}{4} = \frac{1-\cos 2x}{2} = \sin^2 x \quad \boxed{\checkmark}\,.$$

So

$$\begin{aligned}\int_{x=0}^{x=\frac{\pi}{4}} \sin^2 x\,dx &= \left(\frac{x}{2}-\frac{\sin 2x}{4}\right)\Big|_{x=0}^{x=\frac{\pi}{4}} = \left(\frac{\pi}{8}-\frac{\sin\frac{\pi}{2}}{4}\right) - \left(\frac{0}{2}-\frac{\sin 0}{4}\right) = \left(\frac{\pi}{8}-\frac{1}{4}\right) - \left(\frac{0}{2}-\frac{0}{4}\right)\\ &= \frac{\pi}{8}-\frac{1}{4}.\end{aligned}$$

4. Evaluate

$$\int_{x=0}^{x=\frac{\pi}{4}} \sin^2 x\,\cos x\,dx$$

4soln. First do indefinite integral. Let $u=\sin x$ so $du=\cos x dx$. So

$$\int \sin^2 x\,\cos x\,dx \;=\; \int u^2\,du = \frac{u^3}{3}+C = \frac{\sin^3 x}{3}+C\,.$$

Next check indefinite integral:

$$D_x\left(\frac{\sin^3 x}{3}\right) = \frac{1}{3}\,D_x\left((\sin x)^3\right) = \frac{1}{3}\left(3\sin^{3-1} x\right)D_x \sin x = \sin^2 x\cos x \quad \boxed{\checkmark}$$

So

$$\int_{x=0}^{x=\frac{\pi}{4}} \sin^2 x\,\cos x\,dx \;=\; \frac{\sin^3 x}{3}\Big|_{x=0}^{x=\frac{\pi}{4}} \;=\; \frac{1}{3}\left(\sin^3\frac{\pi}{4}-\sin^3 0\right) \;=\; \frac{1}{3}\left(\left(\frac{\sqrt{2}}{2}\right)^3 - 0^3\right) = \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3\,.$$

5. Evaluate

$$\int\limits_{x=0}^{x=1} \frac{1}{x^2+1}\,dx\,.$$

5soln. $\int\limits_{x=0}^{x=1} \frac{1}{x^2+1}\,dx \;= \arctan x\big|_{x=0}^{x=1} = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

6. The integral

$$\int \frac{x}{x^2+1}\,dx$$

can be evaluated the following way.

6soln. $\int \frac{x}{x^2+1}\, dx$ can be evaluated by a simple u-du substitution with $u = x^2+1$.

If $u = x^2+1$, then $\int \frac{x}{x^2+1}\, dx \; = \frac{1}{2}\int \frac{2x\ dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$.

So one can integrate the integral using simple u-du substitution of $u = x^2+1$.

Note that if one uses Trig. Substitution, the initial substitution would be $x = \tan\theta$.

Note that the integrand is already in its Partial Fraction Decomposition.

7. Let $y = p(x)$ be a polynomial of degree 5.

What is the form of the partial fraction decomposition of

$$\frac{p(x)}{(x^2-1)\,(x^2+1)^2}\ ?$$

Here A, B, C, D, E and F are constants.

7soln. $(x^2-1)\,(x^2+1)^2 = (x-1)\,(x+1)\,(x^2+1)^2$ where $x-1$ and $x+1$ are linear terms while x^2+1 is an irreducible quadratic term. Now see the partial fraction handout from class to see that the PDF takes the form $\dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$.

8. Evaluate

$$\int_1^3 \frac{5x^2+3x-2}{x^3+2x^2}\, dx\ .$$

8soln. Partial Fraction Decomposition Problem. As usual,

(1) we first find the indefinite integral,
(2) then check that our indefinite integral is correct by integrating the indefinite integral and making sure we get the integrand,
(3) and then evaluate our definite integral.

- $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A+C \;\Rightarrow\; A=2$. So $\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$

- Check $D_x\left[2\ln|x| + x^{-1} + 3\ln|x+2|\right] = \frac{2}{x} + -1x^{-2} + \frac{3}{x+2}$
$= \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2} = \frac{2x(x+2) - (x+2) + 3x^2}{x^2(x+2)} = \frac{5x^2+3x-2}{x^3+2x^2}$ ✓

- $\left[3\ln|x+2| + 2\ln x + \frac{1}{x}\right]\Big|_{x=1}^{x=3} =$

$\left[3\ln 5 + 2\ln 3 + \frac{1}{3}\right] - \left[3\ln 3 + \underbrace{2\ln 1}_{=0} + 1\right] =$

$3\ln 5 - \ln 3 - \frac{2}{3}$.

9. Evaluate

$$\int_{x=-\infty}^{x=\infty} \frac{1}{1+x^2}\, dx \ .$$

9soln. $\int_{x=-\infty}^{x=\infty} \frac{1}{1+x^2}\, dx \ = \ \pi$. From our textbook, page 506, Example 2.

HISTORICAL BIOGRAPHY
Lejeune Dirichlet
(1805–1859)

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} \\ &= \lim_{a\to-\infty} \tan^{-1} x \Big]_a^0 \\ &= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\ &= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

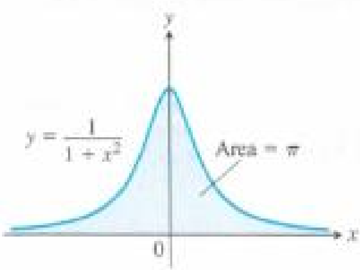


FIGURE 8.15 The area under this curve is finite (Example 2).

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15).

10. Evaluate

$$\int_{-1}^{1} \frac{dx}{x^2} \ .$$

10soln. $\int_{-1}^{1} \frac{dx}{x^2}$ diverges to ∞.

This is an improper integral (the integrand $y = \frac{1}{x^2}$ is continuous on $(-\infty, 0) \cup (0, \infty)$ but is not defined at $x = 0$) and

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[\int_{-1}^{0} \frac{dx}{x^2}\right] + \left[\int_{0}^{1} \frac{dx}{x^2}\right] \ = \ \left[\lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^2}\right] + \left[\lim_{a\to 0^+} \int_{a}^{1} \frac{dx}{x^2}\right] \ .$$

Note

$$\lim_{a\to 0^+}\int_a^1 \frac{dx}{x^2} = \lim_{a\to 0^+}\int_a^1 x^{-2}\,dx = \lim_{a\to 0^+} -x^{-1}\,|_a^1 = \lim_{a\to 0^+}\left[(-1)-\left(\frac{-1}{a}\right)\right] = \lim_{a\to 0^+}\left[\frac{1}{a}-1\right] = \infty.$$

The graph of $y=\frac{1}{x^2}$ is symmetric about the y axis and so $\lim_{b\to 0^-}\int_{-1}^{b}\frac{dx}{x^2} = \infty$; or, you can calculate this limit similarly to the limit we just calculated. So

$$\int_{-1}^{1}\frac{dx}{x^2} = \left[\lim_{b\to 0^-}\int_{-1}^{b}\frac{dx}{x^2}\right] + \left[\lim_{a\to 0^+}\int_a^1\frac{dx}{x^2}\right] = [\infty]+[\infty] = \infty\ .$$

11. Evaluate

$$\int_{-1}^{1}\frac{dx}{x^3}\ .$$

11soln. $\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx$ does not exist but also does not diverge to infinity.

This is an improper integral (the integrand $y=\frac{1}{x^3}$ is continuous on $(-\infty,0)\cup(0,\infty)$ but is not defined at $x=0$) and

$$\int_{-1}^{1}\frac{dx}{x^3} = \left[\int_{-1}^{0}\frac{dx}{x^3}\right] + \left[\int_{0}^{1}\frac{dx}{x^3}\right] = \left[\lim_{b\to 0^-}\int_{-1}^{b}\frac{dx}{x^3}\right] + \left[\lim_{a\to 0^+}\int_a^1\frac{dx}{x^3}\right]$$

- $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$

$$\int_{x=0}^{x=1} x^{-3}\,dx = \lim_{a\to 0^+} \frac{x^{-2}}{-2}\Big|_{x=a}^{x=1} = \frac{1}{2}\lim_{a\to 0^+}\left[\frac{1}{x^2}\right]_{x=1}^{x=a} =$$

$\frac{1}{2}\lim_{x\to 0^+}\left[\frac{1}{a^2}-1\right] = \infty$. Similarly, $\int_{-1}^{0} x^{-3}\,dx = -\infty$.

- $\int_{-1}^{1} x^{-3}\,dx = \int_{-1}^{0} x^{-3}\,dx + \int_0^1 x^{-3}\,dx = -\infty + \infty$ so DNE.