MARK BOX		
PROBLEM	POINTS	
1-25	4x25	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND-IN PART. Hand-in only this part.
  (2) NOT TO HAND-IN PART. This part will not be collected. Take this part home to learn from and to check your answers when the solutions are posted.
- **For the Multiple Choice** problems, circle your answer(s) on the provided chart. No need to show work.
- The MARK BOX above indicates the problems (check that you have them all) along with their points.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1-8.5, 8.7-8.8, 10.1-10.10 and 11.1–11.5 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 4 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 1 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
14	14a	14b	14c	14d	14e					
15	15a	15b	15c	15d	15e					
16	16a	16b	16c	16d	16e					
17	17a	17b	17c	17d	17e					
18	18a	18b	18c	18d	18e					
19	19a	19b	19c	19d	19e					
20	20a	20b	20c	20d	20e					
21	21a	21b	21c	21d	21e					
22	22a	22b	22c	22d	22e					
23	23a	23b	23c	23d	23e					
24	24a	24b	24c	24d	24e					
25	25a	25b	25c	25d	25e					
							4	1	0	0

NOT TO HAND-IN PART
STATEMENT OF MULTIPLE CHOICE PROBLEMS

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.

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1. Evaluate the integral

$$\int_0^1 \frac{x}{x^2+9}\,dx.$$

1soln.

$$\int_{x=0}^{x=1} \frac{x}{x^2+9}\,dx = \frac{1}{2}\int_{u=9}^{u=10} \frac{du}{u} = \frac{1}{2}\ln|u|\,\Big|_{u=9}^{u=10}$$

$$= \frac{1}{2}\ln 10 - \frac{1}{2}\ln 9$$

$u = x^2+9$
$du = 2x\,dx$

$x=0 \Rightarrow u = 9$
$x=1 \Rightarrow u = 10$

2. Evaluate the integral

$$\int_0^4 \frac{x}{x+9}\,dx\ .$$

2soln.

$$\int_0^4 \frac{x}{x+9}\,dx$$

Do not have strictly bigger bottoms so need to do long division.
But it's easy to "fake" long division here:

$$\frac{x}{x+9} = \frac{x+9-9}{x+9} = \frac{x+9}{x+9} - \frac{9}{x+9} = 1 - \frac{9}{x+9}.$$

So

$$\int_0^4 \frac{x}{x+9}\,dx = \int_0^4 \left[1 - \frac{9}{x+9}\right] dx$$

$$= \left[x - 9\ln|x+9|\right]\Big|_{x=0}^{x=4}$$

$$= (4 - 9\ln 13) - (0 - 9\ln 9)$$

$$= 4 - 9\ln 13 + 9\ln 9.$$

3. Evaluate

$$\int_0^{\ln(2\pi)} e^x \cos\left(e^x\right)\, dx$$

3soln. Let $u = e^x$. So $du = e^x dx$. So $\int e^x \cos(e^x)\, dx = \int \cos u\, du = \sin u + C = \sin(e^x) + C$.
Next check indefinite integral: $D_x \sin(e^x) = [\cos(e^x)]\, D_x e^x = [\cos(e^x)]\, e^x$ $\boxed{\checkmark}$.
So $\int_0^{\ln(2\pi)} e^x \cos(e^x)\, dx = \sin e^x \,\Big|_{x=0}^{x=\ln(2\pi)} = \sin e^{\ln(2\pi)} - \sin e^0 = \sin(2\pi) - \sin 1 = 0 - \sin 1 = -\sin 1$

4. Evaluate

$$\int_{x=0}^{x=\frac{3\pi}{2}} e^x \cos x\, dx\ .$$

4soln. Below we show that $\int e^x \cos x\, dx = \frac{e^x(\sin x + \cos x)}{2} + C$.
So $\int_{x=0}^{x=\frac{3\pi}{2}} e^x \cos x\, dx = \frac{e^x(\sin x+\cos x)}{2}\Big|_0^{3\pi/2} = \frac{e^{3\pi/2}(-1)}{2} - \frac{e^0(1)}{2} = \frac{-1-e^{3\pi/2}}{2}$.
To find the indefinite integral, use two integration by parts and the *bring to the other side* idea. For the two integration by parts, put the expontential function with either the u's both times or the dv's both times.

Way # 1

For this way, for each integration by parts, we let the u involve the expontenial function.

$$\begin{aligned} u_1 &= e^{1x} & dv_1 &= \cos 1x\, dx \\ du_1 &= 1e^{1x}\, dx & v_1 &= \frac{1}{1}\sin 1x\ . \end{aligned}$$

So by integration by parts

$$\int e^{1x} \cos 1x\, dx = \frac{1}{1} e^{1x} \sin 1x - \frac{1}{1}\int e^{1x} \sin 1x\, dx\ .$$

Now let

$$\begin{aligned} u_2 &= e^{1x} & dv_2 &= \sin 1x\, dx \\ du_2 &= 1e^{1x}\, dx & v_2 &= \frac{-1}{1}\cos 1x\ . \end{aligned}$$

to get

$$\begin{aligned} \int e^{1x} \cos 1x\, dx &= \frac{1}{1} e^{1x}\sin 1x - \frac{1}{1}\left[\frac{-1}{1} e^{1x}\cos 1x - \frac{-1}{1}\int e^{1x}\cos 1x\, dx\right] \\ &= \frac{1}{1} e^{1x}\sin 1x + \frac{1}{1^2} e^{1x}\cos 1x - \frac{1^2}{1^2}\int e^{1x}\cos 1x\, dx\ . \end{aligned}$$

Now solving for $\int e^{1x}\cos 1x\, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{1^2}{1^2}\right]\int e^{1x}\cos 1x\, dx = \frac{1}{1} e^{1x}\sin 1x + \frac{1}{1^2} e^{1x}\cos 1x + K$$

and so

$$\begin{aligned}\int e^{1x}\cos 1x\;dx &= \left[\frac{1^2}{2}\right]\left(\frac{1}{1}e^{1x}\sin 1x + \frac{1}{1^2}e^{1x}\cos 1x + K\right)\\ &= \frac{1}{2}\,e^{1x}\sin 1x + \frac{1}{2}\,e^{1x}\cos 1x + \left[\frac{K1^2}{2}\right]\\ &= \frac{e^{1x}}{2}\,(1\sin 1x + 1\cos 1x) + \left[\frac{K1^2}{2}\right].\end{aligned}$$

Thus

$$\int e^{1x}\cos 1x\;dx = \boxed{\frac{e^{1x}}{2}\,(1\cos 1x + 1\sin 1x) + C}.$$

$\boxed{\text{Way \# 2}}$

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$u_1 = \cos 1x \qquad dv_1 = e^{1x}dx$$
$$du_1 = -1\sin 1x\,dx \qquad v_1 = \frac{1}{1}\,e^{1x}.$$

So, by integration by parts

$$\int e^{1x}\cos 1x\;dx = \frac{1}{1}e^{1x}\cos 1x - \frac{-1}{1}\int e^{1x}\sin 1x\,dx.$$

Now let

$$u_2 = \sin 1x \qquad dv_2 = e^{1x}dx$$
$$du_2 = 1\cos 1x\,dx \qquad v_2 = \frac{1}{1}\,e^{1x}.$$

to get

$$\begin{aligned}\int e^{1x}\cos 1x\;dx &= \frac{1}{1}e^{1x}\cos 1x + \frac{1}{1}\left[\frac{1}{1}e^{1x}\sin 1x - \frac{1}{1}\int e^{1x}\cos 1x\,dx\right]\\ &= \frac{1}{1}e^{1x}\cos 1x + \frac{1}{1^2}e^{1x}\sin 1x - \frac{1^2}{1^2}\int e^{1x}\cos 1x\,dx.\end{aligned}$$

Now solving for $\int e^{1x}\cos 1x\,dx$ (use the *bring to the other side* idea) we get

$$\left[1+\frac{1^2}{1^2}\right]\int e^{1x}\cos 1x\;dx = \frac{1}{1}e^{1x}\cos 1x + \frac{1}{1^2}e^{1x}\sin 1x + K$$

and so

$$\begin{aligned}\int e^{1x}\cos 1x\;dx &= \left[\frac{1^2}{1^2+1^2}\right]\left(\frac{1}{1}e^{1x}\cos 1x + \frac{1}{1^2}e^{1x}\sin 1x + K\right)\\ &= \frac{1}{2}\,e^{1x}\cos 1x + \frac{1}{2}\,e^{1x}\sin 1x + \left[\frac{K1^2}{1^2+1^2}\right]\\ &= \frac{e^{1x}}{2}\,(1\cos 1x + 1\sin 1x) + \left[\frac{K1^2}{1^2+1^2}\right]\end{aligned}$$

Thus

$$\int e^{1x}\cos 1x\;dx = \boxed{\frac{e^{1x}}{2}\,(1\cos 1x + 1\sin 1x) + C}.$$

Doesn't Work Way

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

5. Evaluate

$$\int_{x=0}^{x=1} xe^x \, d$$

5soln.

$u = x \quad dv = e^x\,dx$

$du = dx \quad v = e^x \quad$ gives

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C. = (x-1)e^x + C$$

• Check $D_x\left[(x-1)e^x\right] = \left[D_x(x-1)\right]e^x + (x-1)\left[D_x e^x\right] = 1e^x + (x-1)e^x = xe^x.$ ✓

• $\displaystyle\int_{x=0}^{x=1} xe^x\,dx = (x-1)e^x\Big|_{x=0}^{x=1} = \left[0 - (-e^0)\right] = e^0 = 1.$

6. Evaluate

$$\int_{x=0}^{x=1} \sin^4 x \, d.$$

6soln.

• From class lecture handout on Trig Integrals

Example 4. $\int \sin^4 x\,dx$

If we try $s = \cos x$ or $t = \sin x$, it will not work (why? $\int \sin^4 x\,dx = -\int \boxed{\sin^3 x}\,[-\sin x\,dx]$).
Here we use the half-angle formulas.

$$\int \sin^4 x\,dx = \int [\sin^2 x]^2\,dx = \int \left[\frac{1-\cos(2x)}{2}\right]^2 dx = \frac{1}{4}\int [1 - 2\cos(2x) + \cos^2(2x)]\,dx$$
$$= \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] dx$$
$$= \frac{3}{8}\int dx - \frac{1}{4}\cdot\int \cos(2x)\,2dx + \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}\int \cos(4x)\,4dx$$
$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

• Check : use half/double -angle formulas.

$$D_x\left[\tfrac{3}{8}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x)\right]$$

$$= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

$$= \left[\frac{1-\cos(2x)}{2}\right] + \frac{1}{4}\left[\frac{1+\cos 2(2x)}{2}\right] + -\frac{1}{4}$$

$$= \sin^2(x) + \frac{1}{4}\cos^2(2x) - \frac{1}{4} = \sin^2(x) + \frac{1}{4}\Big[\underbrace{\cos^2(x)}_{1-\sin^2(x)} - \sin^2(x)\Big]^2 - \frac{1}{4}$$

$$= \sin^2(x) + \frac{1}{4}\left[1 - 2\sin^2(x)\right]^2 - \frac{1}{4}$$

$$= \sin^2(x) + \frac{1}{4}\left[1 - 4\sin^2(x) + 4\sin^4(x)\right] - \frac{1}{4}$$

$$= \sin^2(x) + \frac{1}{4} - \sin^2(x) + \sin^4(x) - \frac{1}{4} = \sin^4(x) \checkmark$$

• $\displaystyle\int_0^1 \sin^4 x\,dx = \left[\frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32}\right]\Bigg|_{x=0}^{x=1} = \left[\frac{3}{8} - \frac{1}{4}\sin 2 + \frac{1}{32}\sin 4\right] - [0].$

7. Evaluate

$$\int_{x=5}^{x=10} \frac{\sqrt{x^2-25}}{x}\, dx$$

AND specify the initial substitution.

7soln.

First, Evaluate $\int \frac{\sqrt{x^2-25}}{x}\,dx$, assuming that $x \geq 5$.

Solution. The integrand involves a radical of the form $\sqrt{x^2-a^2}$ with $a=5$, so from Table 8.4.1 we make the substitution

$$x = 5\sec\theta, \quad 0 \leq \theta < \pi/2$$

$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \text{or} \quad dx = 5\sec\theta\tan\theta\, d\theta$$

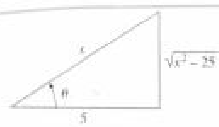
Thus,

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= 5\int \tan^2\theta\, d\theta \qquad \tan\theta \geq 0 \text{ since } 0 \leq \theta < \pi/2$$

$$= 5\int(\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C$$



$x = 5\sec\theta$

Figure 8.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 8.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C \blacktriangleleft$$

• Check $D_x\left[(x^2-25)^{1/2} - 5\sec^{-1}\left(\frac{x}{5}\right)\right]$

$$= \frac{1}{2}(x^2-25)^{-1/2}(2x) - 5\frac{1}{\left|\frac{x}{5}\right|\sqrt{\left(\frac{x}{5}\right)^2-1}}\cdot\frac{1}{5}$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{1}{\frac{x}{5}\sqrt{\frac{x^2}{25}-\frac{25}{25}}} \qquad (\text{know } x \geq 5)$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{25}{x(x^2-25)^{1/2}} = \frac{(x^2-25)}{x(x^2-25)^{1/2}} \cdot \frac{(x^2-25)^{1/2}}{(x^2-25)^{1/2}}$$

$$= \frac{(x^2-25)(x^2-25)^{1/2}}{x(x^2-25)} = \frac{\sqrt{x^2-25}}{x}. \checkmark$$

• $$\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)\Big|_{x=5}^{x=10}$$

$$= \left[\sqrt{100-25} - 5\sec^{-1}2\right] - \left[0 - 5\cancelto{0}{\sec^{-1}1}\right] =$$

$$= \sqrt{75} - 5\cdot\frac{\pi}{3} = 5\sqrt{3} - 5\left(\frac{\pi}{3}\right) = 5\left(\sqrt{3} - \frac{\pi}{3}\right).$$

8. Evaluate

$$\int_{x=1}^{x=3} \frac{5x^2+3x-2}{x^3+2x^2}\,dx\ .$$

8soln.

- $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2)+B(x+2)+Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5=A+C \Rightarrow A=2$. So $\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x+2}\right)dx = 2\ln|x|+\frac{1}{x}+3\ln|x+2|+C$.

- Check $D_x\left[2\ln|x|+x^{-1}+3\ln|x+2|\right] = \frac{2}{x}+-1x^{-2}+\frac{3}{x+2}$
$= \frac{2}{x}-\frac{1}{x^2}+\frac{3}{x+2} = \frac{2x(x+2)-(x+2)+3x^2}{x^2(x+2)} = \frac{5x^2+3x-2}{x^3+2x^2}$ ✓

- $\left[3\ln|x+2|+2\ln x+\frac{1}{x}\right]\Big|_{x=1}^{x=3} =$

$\left[3\ln 5+2\ln 3+\frac{1}{3}\right]-\left[3\ln 3+\underbrace{2\ln 1}_{=0}+1\right] =$

$3\ln 5-\ln 3-\frac{2}{3}$.

9. For which value of p does

$$\int_0^1 \frac{1}{x^p}\,dx = 1.25\ ?$$

9soln.

$\int_0^1 \frac{1}{x^p}\,dx = \int_0^1 x^{-p}\,dx \overset{p\neq 1}{=\!=} \frac{x^{-p+1}}{-p+1}\Big|_{x=0}^{x=1}$

$= \frac{1}{1-p} - 0 = \frac{1}{1-p}$.

Want $\frac{1}{1-p} = 1.25$

$\frac{1}{1-p} = 1\frac{1}{4}$

$\frac{1}{1-p} = \frac{5}{4}$

$1-p = \frac{4}{5}$

$\frac{4}{5} = 1-p$

$p = 1-\frac{4}{5} = \frac{1}{5} = \frac{2}{10} = \boxed{0.2}$

10. Evaluate

$$\int_{x=-1}^{x=1} \frac{1}{x^{2/3}}\, dx \ .$$

10soln. $\int_{x=-1}^{x=1} \frac{1}{x^{2/3}}\, dx \ = \lim_{a\to 0^-} \int_{x=-1}^{x=a} \frac{1}{x^{2/3}}\, dx + \lim_{b\to 0^+} \int_{x=b}^{x=1} \frac{1}{x^{2/3}}\, dx \overset{\text{see below}}{=} 3+3=6.$
Note that $\int_0^1 x^{\frac{-2}{3}}\, dx = \lim_{b\to 0^+} \int_b^1 x^{-\frac{2}{3}}\, dx = \lim_{b\to 0^+} 3x^{\frac{1}{3}}\, |_b^1 = \lim_{b\to 0^+} \left(3 - 3b^{\frac{1}{3}}\right) = 3.$
Similiarly (also by symmetry) $\lim_{a\to 0^-} \int_{-1}^{a} x^{\frac{-2}{3}}\, dx = 3.$

11. Compute

$$\lim_{n\to\infty} \frac{17n^3+4n^2-5}{19n^5}$$

11soln.

divide through by n (to the highest power) $= n^5$

$$\lim_{n\to\infty} \frac{17n^3+4n^2-5}{19n^5+3n^4-8n^3+n^2-8}$$

$$= \lim_{n\to\infty} \frac{\frac{17}{n^2}+\frac{4}{n^3}-\frac{5}{n^5}}{19+\frac{3}{n}-\frac{8}{n^2}+\frac{1}{n^3}-\frac{8}{n^5}}$$

$$= \frac{0+0+0}{19+0-0+0-0} = \frac{0}{19} = 0.$$

12. Compute

$$\lim_{n\to\infty} \frac{\sqrt{9n^4+1}}{17n^2+n+3}$$

12soln.

$$\lim_{n\to\infty} \frac{\sqrt{9n^4+1}}{17n^2+n+3}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt{9n^4+1}}{\sqrt{n^4}}}{\frac{17n^2+n+3}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{9+\frac{1}{n^4}}}{17+\frac{1}{n}+\frac{3}{n^2}}$$

$$= \frac{\sqrt{9}}{17}$$

$$= \frac{3}{17}$$

13. Compute

$$\lim_{n\to\infty} \left(\frac{-1}{2}\right)^n .$$

13soln.

$$\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$$

Geometric Series

$r = -\frac{1}{2}$

$|r| = |-\frac{1}{2}| < 1$

so converges to zero

14. Consider the following two series.

Series A is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Series B is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

14soln.

$\sum_1^\infty \frac{1}{n}$ divgeres. (harmonic series =or= p-series $p = 1 \leq 1$)

=

Now consider

$\sum_1^\infty \frac{(-1)^n}{n}$

Let $u_n = \frac{1}{n}$.

- u_n is decreasing
- $\lim_{n\to\infty} u_n = 0$

So by AST, $\sum \frac{(-1)^n}{n}$ conv.

=

So $\sum_1^\infty \frac{1}{n}$ divg. and $\sum \frac{(-1)^n}{n}$ conv. cond.

15. The series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} .$$

15soln.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)\,(n+7)}}$$

- [ ] ~~absolutely convergent~~ LCT w/ $b_n = \frac{1}{n}$
- [x] conditionally convergent then use AST
- [ ] divergent

Warning: there is a square root in the denominator ... many of you overlooked $\sqrt{\ }$'s on Exams ... see it?.

Abs. Conv? Consider $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right| = \sum \frac{1}{\sqrt{(n+2)(n+7)}}$

Thinking Land $a_n = \frac{1}{\sqrt{(n+2)(n+7)}} \overset{n \text{ big}}{\approx} \frac{1}{\sqrt{n \cdot n}} = \frac{1}{n} = b_n$

LCT. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{(n+2)(n+7)}} = 1$

more details $= \lim_{n\to\infty} \sqrt{\left[\frac{n^2}{(n+2)(n+7)}\right]} = \sqrt{1} = 1 = L$, $0 < L < \infty$

so $\sum b_n$ & $\sum a_n$ do the same thing. $\sum b_n$ divg (harmonic series)

so $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right|$ dvg so not abs. conv.

Cond. Conv? Let's use AST w/ $0 \le u_n = \frac{1}{\sqrt{(n+2)(n+7)}}$

(1) u_n dec., ie $u_n > u_{n+1}$? yes clear.

(2) $\lim_{n\to\infty} \frac{1}{\sqrt{(n+2)(n+7)}} = 0$ ☺

so, by AST, $\sum (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}}$ conv.

10

16. Consider the formal seris $\sum_{n=1}^{\infty} a_n$ where

$$a_n = (-1)^n \frac{(n+1)!}{(2n)!}$$

and let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$

16soln.

$$\sum_{n=8}^{\infty} (-1)^n \frac{(n+1)!}{(2n)!}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

But before you get started .... let

$$a_n = \frac{(n+1)!}{(2n)!} .$$

Then $\boxed{a_{n+1} = \frac{((n+1)+1)!}{(2(n+1))!} = \frac{(n+2)!}{(2n+2)!}}$

Next, simplify $\frac{a_{n+1}}{a_n}$ so that it has NO factorial sign (that is a ! sign) in it.

$\boxed{\frac{a_{n+1}}{a_n} = \frac{n+2}{(2n+1)(2n+2)} \quad \text{or} \quad \frac{n+2}{4n^2+6n+2}}$

Ok, now you should be ready to finish off the problem and check the correct box above.

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)!}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)!} = \frac{(n+2)!}{(n+1)!} \frac{(2n)!}{(2n+2)!} = \frac{(n+1)!\,(n+2)}{(n+1)!} \cdot \frac{(2n)!}{(2n)!\,(2n+1)(2n+2)}$$

Abs. Conv? Consider $\sum \frac{(n+1)!}{(2n)!}$ & use Ratio Test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \overset{\text{above}}{=} \lim_{n\to\infty} \frac{n+2}{(2n+1)(2n+2)} = 0 < 1$$

$\Downarrow$ Ratio Test

conv.

17. Find the sum of the series

$$\sum_{n=10}^{\infty} \frac{3^{n+1}}{4^n}\,.$$

17soln.

$$\sum_{n=10}^{\infty} \frac{3^{n+1}}{4^n} = \sum_{n=10}^{\infty} 3 \cdot \left(\frac{3}{4}\right)^n$$

Let $S_N = \sum_{n=10}^{N} 3 \cdot \left(\frac{3}{4}\right)^n$.

$$1 \; S_N = 3 \left[\left(\frac{3}{4}\right)^{10} + \left(\frac{3}{4}\right)^{11} + \cdots + \left(\frac{3}{4}\right)^{N} \right]$$

$$\frac{3}{4} \; S_N = 3 \left[\qquad \left(\frac{3}{4}\right)^{11} + \cdots + \left(\frac{3}{4}\right)^{N} + \left(\frac{3}{4}\right)^{N+1} \right]$$

$$\left(1 - \frac{3}{4}\right) S_N = 3 \left[\left(\frac{3}{4}\right)^{10} - \left(\frac{3}{4}\right)^{N+1} \right]$$

$$\frac{1}{4} \; S_N = 3 \left[\left(\frac{3}{4}\right)^{10} - \left(\frac{3}{4}\right)^{N+1} \right]$$

$$S_N = 12 \left[\left(\frac{3}{4}\right)^{10} - \left(\frac{3}{4}\right)^{N+1} \right]$$

$$\lim_{N\to\infty} S_N = 12 \cdot \left(\frac{3}{4}\right)^{10}$$

18. What is the LARGEST interval for which the power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

is absolutely convergent?

18soln.

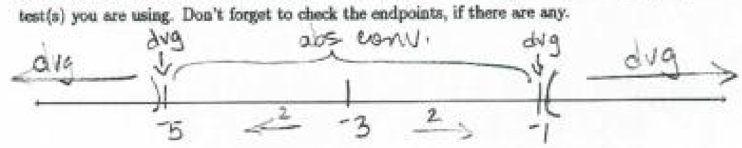
Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

Hint: $(2x+6)^n = [2(x+3)]^n = 2^n (x+3)^n = 2^n (x-(-3))^n$

The center is $x_0 =$ -3 and the radius of convergence is $R =$ 2.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



Ratio Test $\rho = \lim_{n\to\infty} \left| \frac{(2x+6)^{n+1}}{4^{n+1}} \frac{4^n}{(2x+6)^n} \right| = \lim_{n\to\infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$\updownarrow$ or

Root Test. $\rho = \lim_{n\to\infty} \left| \frac{(2x+6)^n}{4^n} \right|^{1/n} = \lim_{n\to\infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$\rho < 1 \iff |2x+6| < 4 \iff 2|x+3| < 4 \iff |x+3| < 2 \iff |x - (-3)| < 2$

endpts. $-3+2 = -1$ and $-3-2 = -5$

Check endpts

$x = -1$: $\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1 = 1+1+1+1+\cdots = \infty$ dvg

$x = -5$ $\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1\cdot 4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n}$

$= \sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+1-1+\cdots$

osc btw -1 & 0 $\Rightarrow$ dvg

19. Suppose that the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is 16. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{2n}$?

19soln.

$\sum c_n x^{2n} = \sum c_n (x^2)^n$ converges when:

$|x^2| < 16$

$|x|^2 < 16$

$|x| < 4$

20. In class we learned that, for each $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} .$$

Use this to find a Taylor expansion for

$$f(x) = x\cos(4x) .$$

center about $x_0 = 0$ (so, Maclaurin series).

20soln.

Since $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, then

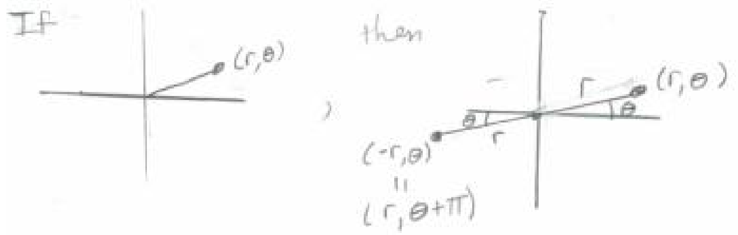
$$\cos(4x) = \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n} x^{2n}}{(2n)!} .$$

$$\therefore \quad F(x) = x\cos(4x) = x \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n+1}}{(2n)!}$$

21. In Polar Coordinates, a point (r, θ) also has which of the following representations?

21soln.



so (r,θ) can be represented as $(-r, \theta+\pi)$.

So can see geometrically. Algebraically, let (r,θ) have cartesian Coordinates (x, y) and $(-r, \theta+\pi)$ " (x_0, y_0). Then

$x_0 = -r\cos(\theta+\pi) = -r(-\cos\theta) = r\cos\theta = x,$

$y_0 = -r\sin(\theta+\pi) = -r(-\sin\theta) = r\sin\theta = y.$

So $(x,y) = (x_0, y_0)$.

22. Find a parameterization for the line segment from $(-1, 2)$ to $(10, -6)$ for $0 \le t \le 1$.

22soln. ans: $x = -1 + 11t$ and $y = 2 - 8t$

$$x(t) = -1 + \left(10 - \left({}^{-}1\right)\right)t = -1 + 11t$$
$$y(t) = 2 + \left({}^{-}6 - 2\right)t \qquad = 2 - 8t.$$

23. Find an equation of the tangent line to the curve at the point corresponding to $t = 11\pi$.

$$x = t\sin t$$
$$y = t\cos t.$$

23soln. ans: $y = \frac{x}{11\pi} - 11\pi$
$(x(11\pi), y(11\pi)) = (0, -11\pi)$

$$\left.\frac{dy}{dx}\right|_{t=11\pi} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=11\pi} = \left.\frac{\cos t - t\sin t}{\sin t + t\cos t}\right|_{t=11\pi} = \frac{-1-0}{0-11\pi} = \frac{1}{11\pi}.$$

So equation of tangent line to curve when $t = 11\pi$ is

$$\left(y - {}^{-}\,11\pi\right) = \frac{1}{11\pi}(x - 0)$$
$$y + 11\pi = \frac{1}{11\pi}x$$
$$y = \frac{1}{11\pi}x - 11\pi.$$

24. Express the polar equation

$$r = 2\sin\theta$$

in Cartesion equations.

24soln.

$r = 2\sin\theta \iff r^2 = 2r\sin\theta$

$\iff x^2 + y^2 = 2y$ $\boxed{\leftarrow \text{ ok, but can you do better? what is it?}}$

$\iff x^2 + y^2 - 2y + 1 = 1$

$\iff x^2 + (y-1)^2 = 1.$ ← circle with $\begin{cases} \text{radius} = 1 \\ \text{center} = (0,1) \end{cases}$

25. Express the area enclosed by

$$r = 5 - 5\sin\theta$$

as on integral.

25soln.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

There are many answers (due to symmetry)

$$A = \frac{1}{2}\int_0^{2\pi} [5 - 5\sin\theta]^2 \, d\theta$$

$$\overset{\text{or}}{=} 2 \cdot \frac{1}{2}\int_{-\pi/2}^{\pi/2} [5 - 5\sin\theta]^2 \, d\theta$$

$$\overset{\text{or}}{=} 2 \cdot \frac{1}{2}\int_0^{\pi/2} [5 - 5\sin\theta]^2 \, d\theta + 2\,\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} [5 - 5\sin\theta]^2 \, d\theta$$

$$\overset{\text{or}}{=} 2 \cdot \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} [5 - 5\sin\theta]^2 \, d\theta + 2 \cdot \frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} [5 - 5\sin\theta]^2 \, d\theta$$

$$\overset{\text{or}}{=} 2 \cdot \frac{1}{2}\int_{\pi/2}^{3\pi/2} [5 - 5\sin\theta]^2 \, d\theta \quad \ldots$$ gosh, we could go on & on.

FYI: $[5 - 5\sin\theta]^2 = 25 - 50\sin^{1}\theta + 25\sin^2\theta = 25\left[\sin^2\theta - 2\sin\theta + 1\right]$

$= 25(1 - \sin\theta)^2$