MARK BOX		
PROBLEM	POINTS	
1-25	4x25	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND-IN PART. Hand-in only this part.
  (2) NOT TO HAND-IN PART. This part will not be collected.
  Take this part home to learn from and to check your answers when the solutions are posted.
- **For the Multiple Choice** problems, circle your answer(s) on the provided chart. No need to show work.
- The MARK BOX above indicates the problems (check that you have them all) along with their points.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13$^{\text{th}}$ ed., ET): §8.1–8.5, 8.7, 8.8, 10.1–10.10, 11.1–11.5 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________________

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 4 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 1 points.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
14	14a	14b	14c	14d	14e					
15	15a	15b	15c	15d	15e					
16	16a	16b	16c	16d	16e					
17	17a	17b	17c	17d	17e					
18	18a	18b	18c	18d	18e					
19	19a	19b	19c	19d	19e					
20	20a	20b	20c	20d	20e					
21	21a	21b	21c	21d	21e					
22	22a	22b	22c	22d	22e					
23	23a	23b	23c	23d	23e					
24	24a	24b	24c	24d	24e					
25	25a	25b	25c	25d	25e					
							4	1	0	0

NOT TO HAND-IN PART
STATEMENT OF MULTIPLE CHOICE PROBLEMS

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.

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1. Evaluate the integral

$$\int_0^{\pi/2} \cos^3 x\,dx.$$

1soln. Break off one cos to form the $du = \cos x dx$ and so $u = \sin x$.

$$\int \cos^3 x\,dx = \int \left(\cos^2 x\right) \cos x\,dx = \int \left(1 - \sin^2 x\right) \cos x\,dx = \int \cos x\,dx - \int \sin^2 x \cos x\,dx = \sin x - \tfrac{1}{3}\sin^3 x + C$$

$$\begin{aligned}\int_0^{\pi/2} \cos^3 x\,dx &= \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} \\ &= \left[\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right] - \left[\sin 0 - \frac{1}{3}\sin^3 0\right] = \left[1 - \frac{1}{3}\right] - [0 - 0] = \frac{2}{3}\end{aligned}$$

2. Evaluate the integral

$$\int_1^2 \frac{8}{x^2 - 2x + 2}\,dx.$$

Hint: complete the square in the denominator.

2soln. $x^2 - 2x + 2 = (x-1)^2 + 1$. So $\frac{8}{x^2-2x+2} = \frac{8}{(x-1)^2+1}$. So let $u = x - 1$.

$u = x - 1 \quad du = dx$

$u = 0$ when $x = 1$, $u = 1$ when $x = 2$

$$\begin{aligned}\int_1^2 \frac{8}{x^2 - 2x + 2}\,dx &= 8\int_0^1 \frac{1}{u^2+1}\,du \\ &= 8\tan^{-1} u\Big]_0^1 = 8\left(\frac{\pi}{4} - 0\right) = 2\pi\end{aligned}$$

3. Evaluate the integral

$$\int_1^2 x \ln x \, dx.$$

3soln. $y = \ln x$ is easy to differentiate but hard to integrate so try parts with $u = \ln x$.

$$u = \ln x,\ du = \frac{dx}{x};\ dv = x\,dx,\ v = \frac{x^2}{2};$$
$$\int_1^2 x \ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

4. Evaluate the integral

$$\int_0^{\pi/8} e^{3x} \cos(4x)\, dx.$$

4soln. First we show that

$$\int e^{3x} \cos 4x \; dx \;=\; \frac{e^{3x}}{25}\,(3\cos 4x + 4\sin 4x) + C\ .$$

We will use two integration by parts and the *bring to the other side* idea. For the two integration by parts, put the expontential function with either the u's both times or the dv's both times.

$\boxed{\text{Way \# 1}}$

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{3x} \qquad\qquad dv_1 = \cos 4x\,dx$$
$$du_1 = 3e^{3x}\,dx \qquad\qquad v_1 = \frac{1}{4}\sin 4x\ .$$

So by integration by parts

$$\int e^{3x} \cos 4x \; dx \;=\; \frac{1}{4}e^{3x}\sin 4x - \frac{3}{4}\int e^{3x}\sin 4x\,dx\ .$$

Now let

$$u_2 = e^{3x} \qquad\qquad dv_2 = \sin 4x\,dx$$
$$du_2 = 3e^{3x}\,dx \qquad\qquad v_2 = \frac{-1}{4}\cos 4x\ .$$

to get

$$\begin{aligned}\int e^{3x} \cos 4x \; dx &= \frac{1}{4}e^{3x}\sin 4x - \frac{3}{4}\left[\frac{-1}{4}e^{3x}\cos 4x - \frac{-3}{4}\int e^{3x}\cos 4x\,dx\right]\\ &= \frac{1}{4}e^{3x}\sin 4x + \frac{3}{4^2}e^{3x}\cos 4x - \frac{3^2}{4^2}\int e^{3x}\cos 4x\,dx\ .\end{aligned}$$

Now solving for $\int e^{3x}\cos 4x\,dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{3^2}{4^2}\right]\int e^{3x}\cos 4x\;dx \;=\; \frac{1}{4}e^{3x}\sin 4x + \frac{3}{4^2}e^{3x}\cos 4x + K$$

and so

$$\begin{aligned}
\int e^{3x} \cos 4x \, dx &= \left[\frac{4^2}{25}\right] \left(\frac{1}{4} e^{3x} \sin 4x + \frac{3}{4^2} e^{3x} \cos 4x + K\right) \\
&= \frac{4}{25}\, e^{3x} \sin 4x + \frac{3}{25}\, e^{3x} \cos 4x + \left[\frac{K4^2}{25}\right] \\
&= \frac{e^{3x}}{25}\, (4 \sin 4x + 3 \cos 4x) + \left[\frac{K4^2}{25}\right] .
\end{aligned}$$

Thus

$$\int e^{3x} \cos 4x \, dx = \boxed{\frac{e^{3x}}{25}\, (3 \cos 4x + 4 \sin 4x) + C} .$$

$\boxed{\text{Way \# 2}}$

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$\begin{aligned}
u_1 &= \cos 4x & dv_1 &= e^{3x} dx \\
du_1 &= -4 \sin 4x \, dx & v_1 &= \frac{1}{3}\, e^{3x} .
\end{aligned}$$

So, by integration by parts

$$\int e^{3x} \cos 4x \, dx = \frac{1}{3} e^{3x} \cos 4x - \frac{-4}{3} \int e^{3x} \sin 4x \, dx .$$

Now let

$$\begin{aligned}
u_2 &= \sin 4x & dv_2 &= e^{3x} dx \\
du_2 &= 4 \cos 4x \, dx & v_2 &= \frac{1}{3}\, e^{3x} .
\end{aligned}$$

to get

$$\begin{aligned}
\int e^{3x} \cos 4x \, dx &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \left[\frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \int e^{3x} \cos 4x \, dx\right] \\
&= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3^2} e^{3x} \sin 4x - \frac{4^2}{3^2} \int e^{3x} \cos 4x \, dx .
\end{aligned}$$

Now solving for $\int e^{3x} \cos 4x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{4^2}{3^2}\right] \int e^{3x} \cos 4x \, dx = \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3^2} e^{3x} \sin 4x + K$$

and so

$$\begin{aligned}
\int e^{3x} \cos 4x \, dx &= \left[\frac{3^2}{3^2 + 4^2}\right] \left(\frac{1}{3} e^{3x} \cos 4x + \frac{4}{3^2} e^{3x} \sin 4x + K\right) \\
&= \frac{3}{25}\, e^{3x} \cos 4x + \frac{4}{25}\, e^{3x} \sin 4x + \left[\frac{K3^2}{3^2 + 4^2}\right] \\
&= \frac{e^{3x}}{25}\, (3 \cos 4x + 4 \sin 4x) + \left[\frac{K3^2}{3^2 + 4^2}\right]
\end{aligned}$$

Thus

$$\int e^{3x} \cos 4x \, dx = \boxed{\frac{e^{3x}}{25}\, (3 \cos 4x + 4 \sin 4x) + C} .$$

Doesn't Work Way

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

Now evaluate the indefinite integral.

So

$$\begin{aligned}
\int_0^{\pi/8} e^{3x} \cos(4x)\, dx &= \left[\frac{e^{3x}}{25}\left(3\cos 4x + 4\sin 4x\right)\right] \Big|_{x=0}^{x=\pi/8} \\
&= \left[\frac{e^{3\pi/8}}{25}\left(3\cos\frac{\pi}{2} + 4\sin\frac{\pi}{2}\right)\right] - \left[\frac{e^0}{25}\left(3\cos 0 + 4\sin 0\right)\right] \\
&= \left[\frac{e^{3\pi/8}}{25}\,(4)\right] - \left[\frac{1}{25}\,(3)\right] = \frac{4e^{3\pi/8} - 3}{25}
\end{aligned}$$

5. Evaluate the integral

$$\int_0^{\sqrt{3}/2} \frac{4x^2\, dx}{(1-x^2)^{3/2}}$$

5soln.

$x = \sin\theta,\ 0 \le \theta \le \frac{\pi}{3},\ dx = \cos\theta\, d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta;$

$$\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{\left(1-x^2\right)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3}\left(\sec^2\theta - 1\right) d\theta = 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

6. Evaluate the integral

$$\int_0^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\, dt$$

6soln. Partial Fraction Decompostion. $t^3 + t = t\left(t^2+1\right) = (t-0)\left(t^2+1\right)$.

$$\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} = \frac{A\left(t^2+1\right) + (Bt+C)\,t}{t^3+t}.$$

So

$$3t^2 + t + 4 = A\left(t^2+1\right) + (Bt+C)\,t\ ,$$

which gives

$$\begin{aligned}
t^2 &: \quad 3 = A + B \\
t^1 &: \quad 1 = \quad C \\
t^0 &: \quad 4 = \quad A
\end{aligned}$$

and so $B = 1 - 3 = 3 - 4 = -1$. Thus

$$\int \frac{3t^2+t+4}{t^3+t}\, dt = \int \frac{4}{t}\, dt + \int \frac{-t+1}{t^2+1}\, dt\ .$$

Note

$$\int_0^{\sqrt{3}} \frac{-t+1}{t^2+1}\,dt = \frac{-1}{2}\int_0^{\sqrt{3}} \frac{2t\,dt}{t^2+1}\,dt + \int_0^{\sqrt{3}} \frac{1}{t^2+1}\,dt = \left[\frac{-1}{2}\ln\left|t^2+1\right| + \tan^{-1} t\right]\Big|_0^{\sqrt{3}}$$

$$= \left[\frac{-\ln 4}{2} + \tan^{-1}\sqrt{3}\right] - \left[\frac{-\ln 1}{2} + \tan^{-1} 0\right] = \left[\frac{-\ln(2^2)}{2} + \frac{\pi}{3}\right] - [0-0] = \frac{\pi}{3} - \ln 2.$$

Also

$$\int_0^{\sqrt{3}} \frac{4}{t}\,dt = 4\lim_{b\to 0+}\int_b^{\sqrt{3}} \frac{dt}{t} = 4\lim_{b\to 0+} \ln|t|\,\Big|_b^{\sqrt{3}} = 4\lim_{b\to 0+}\left[\ln\sqrt{3} - \ln b\right] = \infty.$$

7. Fill in the two blanks. By comparing the improper integral

$$\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$$

to the improper integral ____________________ ,
the Direct Comparison Test (for improper integrals) gives that $\displaystyle\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$ is __________.

7soln.

$\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$. Since for $0 \le t \le \pi, 0 \le \frac{1}{\sqrt{t}+\sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^{\pi} \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

8. Limit of a sequence. Evaluate

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}}$$

8soln.

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty}\left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2} \Rightarrow \text{ converges}$$

9. Limit of a sequence.

$$\lim_{n\to\infty} \frac{\sin n}{n}.$$

9soln.

$\lim_{n\to\infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

10. Find the vaule for b for which

$$e^b + e^{2b} + e^{3b} + e^{4b} + \ldots = 9.$$

10soln. The geometric series $e^b + e^{2b} + e^{3b} + e^{4b} + \ldots = \sum_{n=1}^{\infty} e^{nb} = \sum_{n=1}^{\infty} \left(e^b\right)^n$ has ratio $r = e^b$ and so it converges precisely when $\left|e^b\right| < 1$. Note that

$$\left|e^b\right| < 1 \Longleftrightarrow 0 < e^b < 1 \Longleftrightarrow -\infty < b < 0.$$

So let

$$b < 0 \qquad \text{and so we know that} \qquad \sum_{n=1}^{\infty} \left(e^b\right)^n \qquad \text{converges.}$$

Let $s_n = \sum_{k=1}^{n} \left(e^b\right)^k$. Then

$$\begin{aligned} s_n &= e^b + e^{2b} + e^{3b} + \ldots + e^{(n-1)b} + e^{nb} \\ e^b s_n &= e^{2b} + e^{3b} + e^{4b} + \ldots + e^{nb} + e^{(n+1)b} \end{aligned}$$

$$\left(1 - e^b\right) s_n = e^b - e^{(n+1)b}$$

$$s_n \overset{\text{know } e^b \neq 1}{=} \frac{e^b - \left(e^b\right)^{n+1}}{1 - e^b} \xrightarrow{\text{as } n \to \infty, \text{ using that } \left|e^b\right| < 1} \frac{e^b}{1 - e^b}.$$

Note

$$9 = \frac{e^b}{1 - e^b} \Longleftrightarrow 9\left(1 - e^b\right) = e^b \Longleftrightarrow 9 - 9e^b = e^b \Longleftrightarrow 9 = 10e^b \Longleftrightarrow e^b = \frac{9}{10} \Longleftrightarrow b = \ln \frac{9}{10}.$$

11. Determine the behavior of the series

$$\sum_{n=1}^{\infty} \frac{2^n \, n! \, n!}{(2n)!}.$$

11soln.

converges by the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n n! n!} = \lim_{n\to\infty} \frac{2(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$

12. The series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1},$$

by the <u>Direct</u> Comparison Test,

12soln.

converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series

$n^2 + 1 > n^2 \Rightarrow n^2 + 1 > \sqrt{n} \cdot n^{3/2} \Rightarrow \frac{n^2+1}{\sqrt{n}} > n^{3/2} \Rightarrow \frac{\sqrt{n}}{n^2+1} < \frac{1}{n^{3/2}}$ or use Limit Comparison Test with $\frac{1}{n^{3/2}}$.

13. The series

$$\sum_{n=1}^{\infty} \frac{1}{1 + \ln n},$$

by the <u>Limit</u> Comparison Test,

diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{1+\ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{1+\ln n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} n = \infty$$

13soln.

14. The series

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n},$$

is

14soln.

converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x)+1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test, $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x}\right) dx$

$= \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \lim_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges

15. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, 3^{2n} \, (x-2)^n}{3n}$$

Recall that the interval of convergence is the set of x for which the power series converges, either absolutely or conditionally.

15soln.

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(-1)^{n+1} 3^{2n+2} (x-2)^{n+1}}{3(n+1)} \cdot \frac{3n}{(-1)^n 3^{2n} (x-2)^n}\right| < 1 \Rightarrow |x-2| \lim_{n\to\infty} \frac{9n}{n+1} = 9|x-2| < 1 \Rightarrow \frac{17}{9} < x < \frac{19}{9}$;

when $x = \frac{17}{9}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}}{3n} \left(-\frac{1}{9}\right)^n = \sum_{n=1}^{\infty} \frac{1}{3n}$, a divergent series; when $x = \frac{19}{9}$ we have

$\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}}{3n} \left(\frac{1}{9}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$, a conditionally convergent series.

(a) the radius is $\frac{1}{9}$; the interval of convergence is $\frac{17}{9} < x \leq \frac{19}{9}$

(b) the interval of absolute convergence is $\frac{17}{9} < x < \frac{19}{9}$

(c) the series converges conditionally at $x = \frac{19}{9}$

16. Using the geometric series, find a power series representation about (i.e., centered at) $x = 5$ for the function

$$g(x) = \frac{3}{x-2}$$

and indicate when the representation is valid.

16soln.

$$g(x) = \frac{3}{x-2} = \frac{3}{3-[-(x-5)]} = \frac{1}{1-\left[-\left(\frac{x-5}{3}\right)\right]} = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n (x-5)^n,$$ which converges for

$\left|\frac{x-5}{3}\right| < 1$ or $2 < x < 8$.

17. Find the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$.

17soln. The computations below show that the 3rd order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$ is $p_3(x) = 5 + 3(x-1) + 9(x-1)^2 + 10(x-1)^3$.

we were given $x_0 = 1$			
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$\frac{f^{(n)}(x_0)}{n!}$
0	$x^5 - x^2 + 5$	5	$\frac{5}{0!} = \frac{5}{1} = 5$
1	$5x^4 - 2x$	$5 - 2 = 3$	$\frac{3}{1!} = \frac{3}{1} = 3$
2	$5 \cdot 4x^3 - 2$	$20 - 2 = 18$	$\frac{18}{2!} = \frac{18}{2} = 9$
3	$5 \cdot 4 \cdot 3$	$(5)(4)(3)$	$\frac{(5)(4)(3)}{3!} = \frac{(5)(4)(3)}{(3)(2)} = \frac{(5)(4)}{2} = 10$

18. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) \;=\; \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$. Hint. Start with the Taylor series expansion

$$\frac{1}{1-x} \;=\; \sum_{k=0}^{\infty} x^k \qquad \text{valid for } |x| < 1$$

and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} \;=\; (-1)(1-x)^{-2}\, D_x(1-x) \;=\; (-1)(1-x)^{-2}\,(-1) \;=\; (1-x)^{-2}\,.$$

18soln.

Start with Geometric Series and take Derivatives as many times as need.
Geometric Series is valid when $|x|<1$ so resulting power series expansions will also be valid when $|x|<1$.

Geometric Series $\Rightarrow$ $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \xRightarrow{D_x} (1-x)^{-2} = \sum_{k=1}^{\infty} k\, x^{k-1}$

$\xRightarrow{D_x} 2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1)\, x^{k-2} \xRightarrow{D_x} 2\cdot 3(1-x)^{-4} = \sum_{k=3}^{\infty} k(k-1)(k-2)\, x^{k-3}$

So

$$(1-x)^{-4} = \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{6}\, x^{k-3} = \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6}\, x^{n}$$

$\boxed{\text{let } k-3=n} \Rightarrow k = n+3$

19. Consider the function

$$f(x) = e^{-x}\,.$$

The 5th order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_5(x) \;=\; \sum_{n=0}^{5} \frac{(-x)^n}{n!} \;=\; 1 \;-\; x \;+\; \frac{x^2}{2!} \;-\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;-\; \frac{x^5}{5!}\,.$$

The 5th order Remainder term $R_5(x)$ is defined by $R_5(x) = f(x) - P_5(x)$ and so $e^{-x} \approx P_5(x)$ where the approximation is within an error of $|R_5(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_5(x)|$ that is valid for each $x \in (-1, 3)$.

19soln.

For each $x \in (-1,3)$, there exists $c \in (-1,3)$ so that

$$|R_5(x)| = \left| \frac{f^{(6)}(c)}{6!} (x-0)^6 \right| = \frac{1}{6!} e^{-c} |x|^6 \le \frac{1}{6!} e^{-(-1)} 3^6$$

20. Find an equation for the line tangent to the curve parameterized by

$$x = 2t^2 + 3$$
$$y = t^4$$

at the point defined by the value $t = -1$.

20soln.

$t = -1 \Rightarrow x = 5,\ \ y = 1;\ \frac{dx}{dt} = 4t,\ \frac{dy}{dt} = 4t^3 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{4t} = t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = (-1)^2 = 1;$ tangent line is

$$y - 1 = 1\cdot(x-5) \text{ or } y = x - 4;$$

21. Find the Cartesian coordinates of the point with polar coordinates

$$\left(-3, \frac{5\pi}{6}\right)$$

21soln.

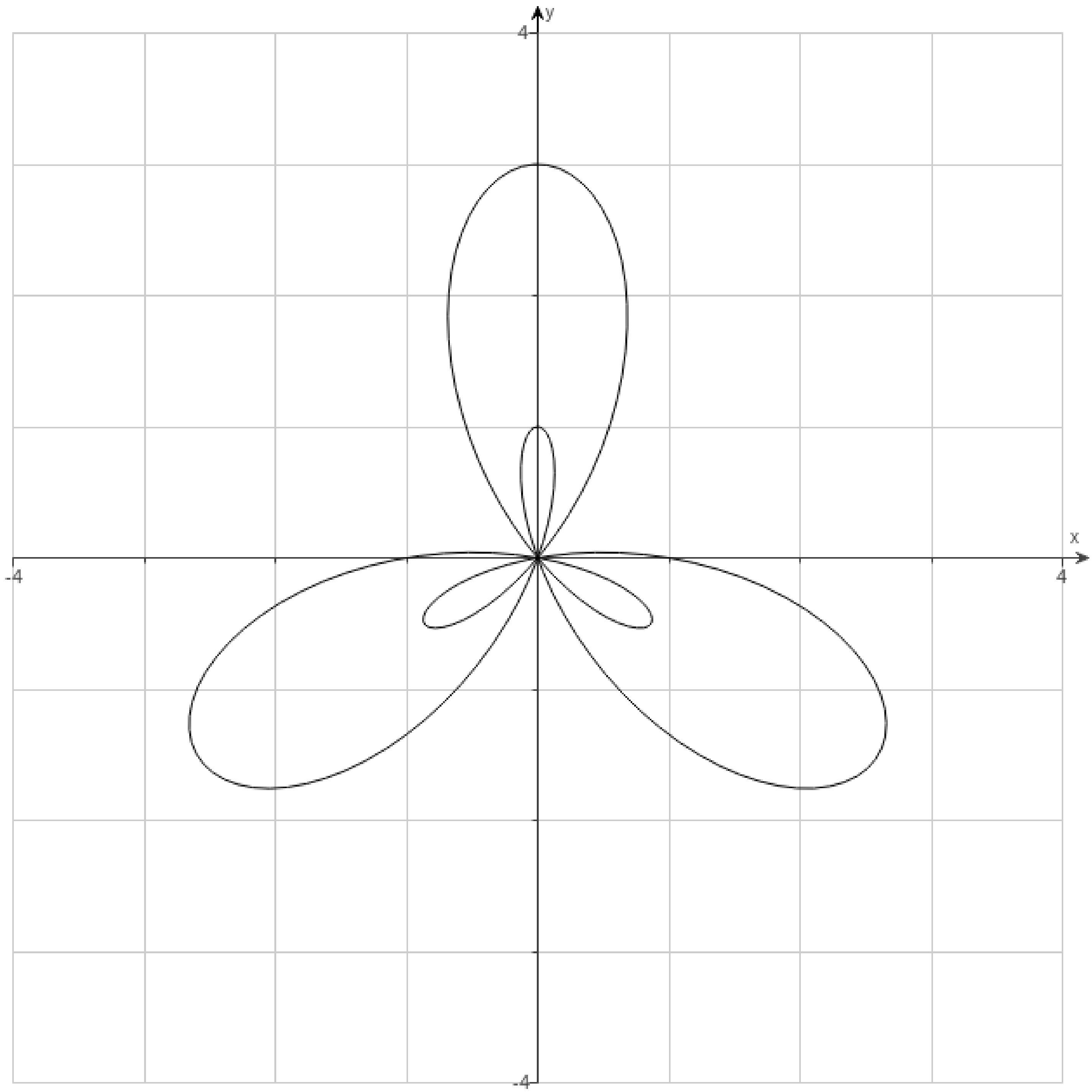
$x = -3\cos\frac{5\pi}{6} = \frac{3\sqrt{3}}{2}, y = -3\sin\frac{5\pi}{6} = -\frac{3}{2} \Rightarrow$ Cartesian coordinates are $\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

22. A parametrization of a circle with center at $(0, 0)$ and radius 1, which is traced out twice in the clockwise direction is

22soln. $x(t) = \cos t$ and $y(t) = -\sin t$ for $0 \le t \le 4\pi$. Note $[x(t)]^2 + [y(t)]^2 = 1$ so the puffo is running around a circle with center $(0, 0)$ and radius 1. The negative on the y makes the tracing go clockwise while $0 \le t \le 4\pi$ traces the circle twice.

23. Choose the correct graph of the polar equation $r = 1 - 2\sin(3\theta)$.

23soln.



- $\left(\frac{1}{4}\right)$ (period of $y = \sin(3x)$) $= \left(\frac{1}{4}\right)\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$ and
- $r = 0 \iff 1 - 2\sin(3\theta) = 0 \iff \sin(3\theta) = \frac{1}{2}$

$$\iff \left[3\theta \in \left\{\frac{\pi}{6} + 2\pi k \colon k \in \mathbb{N}\right\} \cup \left\{\frac{5\pi}{6} + 2\pi k \colon k \in \mathbb{N}\right\}\right]$$

$$\iff \left[\theta \in \left\{\frac{\pi}{18} + \frac{12\pi k}{18} \colon k \in \mathbb{N}\right\} \cup \left\{\frac{5\pi}{18} + \frac{12\pi k}{18} \colon k \in \mathbb{N}\right\}\right]$$

	θ	θ	3θ	$\sin(3\theta)$	-2 $\sin(3\theta)$	$r = 1 - 2\sin(3\theta)$
1	$\frac{0\pi}{6} \to \frac{1\pi}{6}$	$0 \to \frac{\pi}{6}$	$0 \to \frac{\pi}{2}$	$0 \to 1$	$0 \to -2$	$1 \to -1$
2	$\frac{1\pi}{6} \to \frac{2\pi}{6}$	$\frac{\pi}{6} \to \frac{\pi}{3}$	$\frac{\pi}{2} \to \pi$	$1 \to 0$	$-2 \to 0$	$-1 \to 1$
3	$\frac{2\pi}{6} \to \frac{3\pi}{6}$	$\frac{\pi}{3} \to \frac{\pi}{2}$	$\pi \to \frac{3\pi}{2}$	$0 \to -1$	$0 \to 2$	$1 \to 3$
4	$\frac{3\pi}{6} \to \frac{4\pi}{6}$	$\frac{\pi}{2} \to \frac{2\pi}{3}$	$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$2 \to 0$	$3 \to 1$
5	$\frac{4\pi}{6} \to \frac{5\pi}{6}$	$\frac{2\pi}{3} \to \frac{5\pi}{6}$		$0 \to 1$	$0 \to -2$	$1 \to -1$
6	$\frac{5\pi}{6} \to \frac{6\pi}{6}$	$\frac{5\pi}{6} \to \pi$		$1 \to 0$	$-2 \to 0$	$-1 \to 1$
7	$\frac{6\pi}{6} \to \frac{7\pi}{6}$	$\pi \to \frac{7\pi}{6}$		$0 \to -1$	$0 \to 2$	$1 \to 3$
8	$\frac{7\pi}{6} \to \frac{8\pi}{6}$	$\frac{7\pi}{6} \to \frac{4\pi}{3}$		$-1 \to 0$	$2 \to 0$	$3 \to 1$
9	$\frac{8\pi}{6} \to \frac{9\pi}{6}$	$\frac{4\pi}{3} \to \frac{3\pi}{2}$		$0 \to 1$	$0 \to -2$	$1 \to -1$
10	$\frac{9\pi}{6} \to \frac{10\pi}{6}$	$\frac{3\pi}{2} \to \frac{5\pi}{3}$		$1 \to 0$	$-2 \to 0$	$-1 \to 1$
11	$\frac{10\pi}{6} \to \frac{11\pi}{6}$	$\frac{5\pi}{3} \to \frac{11\pi}{6}$		$0 \to -1$	$0 \to 2$	$1 \to 3$
12	$\frac{11\pi}{6} \to \frac{12\pi}{6}$	$\frac{11\pi}{6} \to 2\pi$		$-1 \to 0$	$2 \to 0$	$3 \to 1$
note $r = 0$ when $\theta = \frac{\pi}{18} \in \left[0, \frac{\pi}{6}\right]$						
1.1		$0 \to \frac{\pi}{18}$	$0 \to \frac{\pi}{6}$	$0 \to \frac{1}{2}$	$0 \to -1$	$1 \to 0$
1.2		$\frac{\pi}{18} \to \frac{\pi}{6}$	$\frac{\pi}{6} \to \frac{\pi}{2}$	$\frac{1}{2} \to 1$	$-1 \to -2$	$0 \to -1$
note $r = 0$ when $\theta = \frac{5\pi}{18} \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$						
2.1		$\frac{\pi}{6} \to \frac{5\pi}{18}$	$\frac{\pi}{2} \to \frac{5\pi}{6}$	$1 \to \frac{1}{2}$	$-2 \to -1$	$-1 \to 0$
2.2		$\frac{5\pi}{18} \to \frac{\pi}{3}$	$\frac{5\pi}{6} \to \pi$	$\frac{1}{2} \to 0$	$-1 \to 0$	$0 \to 1$
note $r = 0$ when $\theta = \frac{13\pi}{18} \in \left[\frac{2\pi}{3}, \frac{5\pi}{6}\right]$						
5.1		$\frac{2\pi}{3} \to \frac{13\pi}{18}$	$2\pi \to \frac{13\pi}{6}$	$0 \to \frac{1}{2}$	$0 \to -1$	$1 \to 0$
5.2		$\frac{13\pi}{18} \to \frac{5\pi}{6}$	$\frac{13\pi}{18} \to \frac{5\pi}{2}$	$\frac{1}{2} \to 1$	$-1 \to -2$	$0 \to -1$
note $r = 0$ when $\theta = \frac{17\pi}{18} \in \left[\frac{5\pi}{6}, \pi\right]$						
6.1		$\frac{5\pi}{6} \to \frac{17\pi}{18}$	$\frac{5\pi}{2} \to \frac{17\pi}{6}$	$1 \to \frac{1}{2}$	$-2 \to -1$	$-1 \to 0$
6.2		$\frac{17\pi}{18} \to \pi$	$\frac{17\pi}{6} \to 3\pi$	$\frac{1}{2} \to 0$	$-1 \to 0$	$0 \to 1$

24. Express the area, as an integral (or as integrals) with respect to θ, of the region that (described using polar equations) lies inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

- The desired area is $\displaystyle\int_{\pi/6}^{\pi/2} (3\sin\theta)^2 \, d\theta - \int_{\pi/6}^{\pi/2} (1 + \sin\theta)^2 \, d\theta$
- $\left(\frac{1}{4}\right)$ (period of $y = 3\sin x$ and $y = 1 + \sin x$) $= \left(\frac{1}{4}\right)\left(\frac{2\pi}{1}\right) = \frac{\pi}{2}$

	θ	$\sin\theta$	$r = 3\sin\theta$	$r = 1+\sin\theta$
1	$0 \to \frac{\pi}{2}$	$0 \to 1$	$0 \to 3$	$1 \to 2$
2	$\frac{\pi}{2} \to \pi$	$1 \to 0$	$3 \to 0$	$2 \to 1$
3	$\pi \to \frac{3\pi}{2}$	$0 \to -1$	$0 \to -3$	$1 \to 0$
4	$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$-3 \to 0$	$0 \to 1$

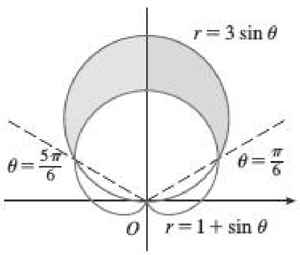


FIGURE 5

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

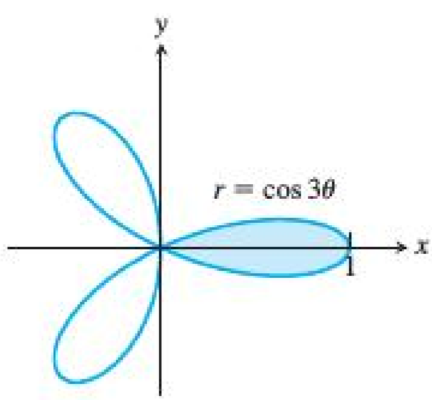
SOLUTION The cardioid (see Example 7 in Section 10.3) and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3\sin\theta = 1 + \sin\theta$, which gives $\sin\theta = \frac{1}{2}$, so $\theta = \pi/6, 5\pi/6$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$. Thus

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2\,d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta)\,d\theta\right]$$

25. Below is a graph of the (polar) equation $r = \cos(3\theta)$, with one leaf shaded in of this three-leaved rose. Express the arc length of this ONE leaf as an integral with respect to θ.



- The Arc Length is $2\displaystyle\int_0^{\pi/6} \sqrt{\cos^2(3\theta) + 9\sin^2(3\theta)}\ d\theta$
- $\left(\frac{1}{4}\right)$ (period of $y = \cos(3x)$) $= \left(\frac{1}{4}\right)\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}$

	θ	θ	3θ	$r = \cos(3\theta)$
1	$\frac{0\pi}{6} \to \frac{1\pi}{6}$	$0 \to \frac{\pi}{6}$	$0 \to \frac{\pi}{2}$	$1 \to 0$
2	$\frac{1\pi}{6} \to \frac{2\pi}{6}$	$\frac{\pi}{6} \to \frac{\pi}{3}$	$\frac{\pi}{2} \to \pi$	$0 \to -1$
3	$\frac{2\pi}{6} \to \frac{3\pi}{6}$	$\frac{\pi}{3} \to \frac{\pi}{2}$	$\pi \to \frac{3\pi}{2}$	$-1 \to 0$
4	$\frac{3\pi}{6} \to \frac{4\pi}{6}$	$\frac{\pi}{2} \to \frac{2\pi}{3}$	$\frac{3\pi}{2} \to 2\pi$	$0 \to 1$
5	$\frac{4\pi}{6} \to \frac{5\pi}{6}$	$\frac{2\pi}{3} \to \frac{5\pi}{6}$		$1 \to 0$
6	$\frac{5\pi}{6} \to \frac{6\pi}{6}$	$\frac{5\pi}{6} \to \pi$		$0 \to -1$
7	$\frac{6\pi}{6} \to \frac{7\pi}{6}$	$\pi \to \frac{7\pi}{6}$		$-1 \to 0$
8	$\frac{7\pi}{6} \to \frac{8\pi}{6}$	$\frac{7\pi}{6} \to \frac{4\pi}{3}$		$0 \to 1$
9	$\frac{8\pi}{6} \to \frac{9\pi}{6}$	$\frac{4\pi}{3} \to \frac{3\pi}{2}$		$1 \to 0$
10	$\frac{9\pi}{6} \to \frac{10\pi}{6}$	$\frac{3\pi}{2} \to \frac{5\pi}{3}$		$0 \to -1$
11	$\frac{10\pi}{6} \to \frac{11\pi}{6}$	$\frac{5\pi}{3} \to \frac{11\pi}{6}$		$-1 \to 0$
12	$\frac{11\pi}{6} \to \frac{12\pi}{6}$	$\frac{11\pi}{6} \to 2\pi$		$0 \to 1$