MARK BOX		
PROBLEM	POINTS	
0	10	
1	12	
2	10	
3-14	48=12x4	
15	10	
16	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- The MARK BOX above indicates the problems along with their points. Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.1-10.6 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0.1. For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums, so $s_n = \sum_{k=1}^{n} a_k$. By definition, the formal series $\sum a_n$ converges if and only if

> the $\lim_{n\to\infty} s_n$ converges (to a finite number). [also ok: the $\lim_{n\to\infty} s_n$ exists (in $\mathbb{R}$)]

0.2. ***p*-series**. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.

0.3. **Geometric Series**. Fill in the boxes with the proper range of $r \in \mathbb{R}$.

- The series $\sum r^n$ converges if and only if r satisfies $|r| < 1$.

0.4. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If $0 \le a_n \le c_n$ (only $a_n \le c_n$ is also ok b/c given $a_n \ge 0$) when $n \ge 17$ and $\sum c_n$ converges, then $\sum a_n$ converges.
- If $0 \le d_n \le a_n$ (need $0 \le d_n$ part here) when $n \ge 17$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.5. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then [$\sum b_n$ converges $\iff$ $\sum a_n$ converges].
- If $L = 0$, then [$\sum b_n$ converges $\implies$ $\sum a_n$ converges] .
- If $L = \infty$, then [$\sum b_n$ diverges $\implies$ $\sum a_n$ diverges] .

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.6. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

- If $\rho > 1$, then $\sum a_n$ diverges .
- If $\rho < 1$, then $\sum a_n$ converges absolutely .

Scoring this page: A problem with precisely one answer marked and the answer is correct, 1 point. All other cases, 0 points.

1. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true.

On the next 3, think of the n^{th}-term test and what if $a_n = \frac{1}{n}$		
(T)	F	If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
T	(F)	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
(T)	F	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
On the next 5, think of AC vs. CC vs. Divergent. Examples from Problem 2 might be helpful.		
(T)	F	A series $\sum a_n$ is precisely one of the following: absolutely convergent, conditionally convergent, divergent.
(T)	F	If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ is either absolutely convergent or divergent.
T	(F)	If $\sum \lvert a_n \rvert$ diverges, then $\sum a_n$ diverges.
(T)	F	If $\sum \lvert a_n \rvert$ converges, then $\sum a_n$ converges.
(T)	F	If $\sum\ a_n$ diverges, then $\sum \lvert a_n \rvert$ diverges.
On the next 2, think of a Theorem from class and what if $b_n = -a_n$.		
(T)	F	If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum (a_n + b_n)$ converges.
T	(F)	If $\sum (a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.
On the next 2, think of a Theorem from class and what if $f(x) = \sin(\pi x)$.		
T	(F)	If a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies that $\lim_{n\to\infty} a_n = L$ and $f\colon [0,\infty) \to \mathbb{R}$ is a function satisfying that $f(n) = a_n$ for each natural number n, then $\lim_{x\to\infty} f(x) = L$.
(T)	F	If a function $f\colon [0,\infty) \to \mathbb{R}$ satisfies that $\lim_{x\to\infty} f(x) = L$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying that $f(n) = a_n$ for each natural number n, that $\lim_{n\to\infty} a_n = L$.

2. Circle the behavior of the given series.

Series	absolutely convergent	conditionally convergent	divergent
$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$	AC	CC	(DVG)
$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$	AC	(CC)	DVG
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	AC	CC	(DVG)
$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	AC	(CC)	DVG
$\sum_{n=1}^{\infty} \frac{1}{n}$	AC	CC	(DVG)
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$	AC	(CC)	DVG
$\sum_{n=1}^{\infty} \frac{1}{n^2}$	(AC)	CC	DVG
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	(AC)	CC	DVG
$\sum_{n=1}^{\infty} \frac{1}{e^n}$	(AC)	CC	DVG
$\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$	(AC)	CC	DV

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 4 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 1 points.
  - ∗ All other cases, 0 points.
- Fill in the “number of solutions circled” column.

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
14	14a	14b	14c	14d	14e					

4	1	0	0

15. Let

$$a_n = \frac{(n!)^2\, 3^n}{(2n+1)!}$$

15.1. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\frac{a_{n+1}}{a_n} = \frac{3\,(n+1)^2}{(2n+2)\ (2n+3)} \qquad \text{answer may vary, eg also ok is} \qquad \frac{3n^2+6n+3}{4n^2+10n+6}$$

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2\ 3^{n+1}}{(2\,(n+1)+1)!} \cdot \frac{(2n+1)!}{(n!)^2\,3^n} = \frac{3^{n+1}}{3^n} \cdot \left[\frac{(n+1)!}{n!}\right]^2 \cdot \frac{(2n+1)!}{(2n+3)!}$$
$$= \frac{(3^n)\,(3)}{3^n} \cdot \left[\frac{n!\ (n+1)}{n!}\right]^2 \cdot \frac{(2n+1)!}{(2n+1)!\ (2n+2)\ (2n+3)} = \frac{3\,(n+1)^2}{(2n+2)\ (2n+3)}.$$

15.2. Carefully justify the behavior of the series below the choice-boxes and then check the correct choice-box. Be sure to clearly explain your logic and specify which test(s) you are using. You may use part 15.1.

$$\sum_{n=1}^{\infty} (-1)^n\ \frac{(n!)^2\,3^n}{(2n+1)!}$$

- [X] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

converges absolutely by the Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!(n+1)!3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n!n!3^n} = \lim_{n\to\infty} \frac{(n+1)^2\,3}{(2n+2)(2n+3)} = \frac{3}{4} < 1$

16. Carefully justify the behavior of the series below the choice-boxes and then check the correct choice-box. Be sure to clearly explain your logic and specify which test(s) you are using.

$$\sum_{n=1}^{\infty} (-1)^n\ \frac{\ln n}{n - \ln n}$$

- [ ] absolutely convergent
- [X] conditionally convergent
- [ ] divergent

Outline of Soln:

converges conditionally since $f(x) = \frac{\ln x}{x - \ln x} \Rightarrow f'(x) = \frac{\left(\frac{1}{x}\right)(x-\ln x) - (\ln x)\left(1-\frac{1}{x}\right)}{(x-\ln x)^2} = \frac{1 - \left(\frac{\ln x}{x}\right) - \ln x + \left(\frac{\ln x}{x}\right)}{(x-\ln x)^2} = \frac{1-\ln x}{(x-\ln x)^2} < 0$

$\Rightarrow u_n \geq u_{n+1} > 0$ when $n > e$ and $\lim_{n\to\infty} \frac{\ln n}{n - \ln n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1-\left(\frac{1}{n}\right)} = 0 \Rightarrow$ convergence; but $n - \ln n < n \Rightarrow \frac{1}{n-\ln n} > \frac{1}{n}$

$\Rightarrow \frac{\ln n}{n - \ln n} > \frac{1}{n}$ so that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n-\ln n}$ diverges by the Direct Comparison Test

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

3. Evaluate

$$\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3}\ .$$

3soln.

$$\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^4}\right)-5}{1+\left(\frac{8}{n}\right)} = -5:$$

4. Evaluate

$$\lim_{n\to\infty} \frac{\sin n}{n}.$$

4soln.

$\lim_{n\to\infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

5. Consider the formal series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

5soln.

$s_k = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{k-1}-\frac{1}{k}\right)+\left(\frac{1}{k}-\frac{1}{k+1}\right)=1-\frac{1}{k+1} \Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)=1,$
series converges to 1

6. The series

$$\sum_{n=17}^{\infty} \frac{1}{n\ln n}$$

is

6soln. Note that the function $f(x) = \frac{1}{x\ln x}$ satifies the conditions os the integral test ($a_n = f(n)$, positive, continuous, decreasing) on the intervial $[2,\infty)$. Let's evaluate the improper integral $\int_2^\infty \frac{1}{x\ln x}\,dx$.

$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \lim_{b\to\infty}\left[\ln(\ln b)-\ln(\ln 2)\right] = \infty,$ so the improper integral diverges

So, by the integral test, $\sum_{n=17}^{\infty} \frac{1}{n\ln n}$ also diverges to ∞.
So the series $\sum_{n=17}^{\infty} \frac{1}{n\ln n}$ is divergent by the Integral Test.

7. The series

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}$$

is

7soln. Compare with $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a convergent geometric series, since $|r| = \left|\frac{1}{3}\right| < 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n \cdot 3^n \geq 3^n \Rightarrow \frac{1}{n \cdot 3^n} \leq \frac{1}{3^n}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$ converges.

So the series $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ is convergent by the Direct Comparison Test using for comparison $\dfrac{1}{3^n}$.

8. The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$$

is

8soln. The series $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ is convergent by the Limit Comparison Test using for comparison $\dfrac{1}{n^{3/2}}$. Since

$$\lim_{n\to\infty} \frac{\left(\frac{n+1}{n^2\sqrt{n}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = 1$$

and $0 < 1 < \infty$ and $\sum \frac{1}{n^{3/2}}$ converges (p-series, $p = 3/2 > 1$), the LCT tells us that $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ also converges.

9. The series

$$\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\left[\ln(n+1)\right]}$$

is

9soln.

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)\cdot 5^{n+1}}{(2(n+1)+3)\ln((n+1)+1)}}{\frac{n\cdot 5^n}{(2n+3)\ln(n+1)}} \right| = \lim_{n\to\infty} \left(\frac{(n+1)\cdot 5^n \cdot 5}{(2n+5)\ln(n+2)} \cdot \frac{(2n+3)\ln(n+1)}{n\cdot 5^n} \right) = \lim_{n\to\infty} \left(\frac{5(n+1)\cdot(2n+3)}{n(2n+5)} \cdot \frac{\ln(n+1)}{\ln(n+2)} \right)$$

$$= \lim_{n\to\infty} \left(\frac{10n^2+25n+15}{2n^2+5n} \right) \cdot \lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n+2)} \right) = \lim_{n\to\infty} \left(\frac{20n+25}{4n+5} \right) \cdot \lim_{n\to\infty} \left(\frac{\frac{1}{n+1}}{\frac{1}{n+2}} \right) = \lim_{n\to\infty} \left(\frac{20}{4} \right) \cdot \lim_{n\to\infty} \left(\frac{n+2}{n+1} \right)$$

$$= 5 \cdot \lim_{n\to\infty} \left(\frac{1}{1} \right) = 5 \cdot 1 = 5 > 1 \Rightarrow \sum_{n=2}^{\infty} \frac{n\cdot 5^n}{(2n+3)\ln(n+1)} \text{ diverges}$$

10. The series

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$$

is

10soln. The series $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$ diverges by the Root Test with $\rho = \infty$.

diverges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty > 1$

If $n \geq 3$ then

$$\left[\frac{(n!)^n}{(n^n)^2}\right]^{1/n} = \left[\frac{(n!)^n}{n^{2n}}\right]^{1/n} = \left[\frac{(n!)^n}{(n^2)^n}\right]^{1/n} = \frac{(n!)}{(n^2)} = \frac{(n-2)!\,(n-1)\,n}{n\cdot n} = (n-2)!\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)$$
$$= (n-2)!\left(1-\frac{1}{n}\right) \xrightarrow{n\to\infty} (\infty)\,(1)\,.$$

11. What is the smallest integer N such that the Alternating Series Estimate/Remainder Theorem guarentees that

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2}\right| \leq 0.05?$$

Note that $0.05 = \dfrac{0.05}{1.0000} = \dfrac{5}{100} = \dfrac{1}{20}$.

11soln. Note that $0 \leq \frac{1}{n^2} \searrow 0$ so the AST applies and we know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges. So by the Alternating Series Estimate/Remainder Theorem

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2}\right| \leq \frac{1}{(N+1)^2}.$$

So

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2}\right| \overset{\text{have}}{\leq} \frac{1}{(N+1)^2} \overset{\text{want}}{\leq} \frac{1}{20}.$$

Note

$$\left[\frac{1}{(N+1)^2} \leq \frac{1}{20}\right] \quad \Leftrightarrow \quad \left[20 \leq (N+1)^2\right].$$

Oooooooohhhh … we can do this without a calculator since

$$20 \not\leq 4^2 = (3+1)^2 \quad \text{but} \quad 20 \leq 5^2 = (4+1)^2\,.$$

So the smallest integer N such that the Alternating Series Estimate/Remainder Theorem guarentees that $\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \sum_{n=1}^{N} \frac{(-1)^n}{n^2}\right| \leq 0.05$ is $N = 4$.

12. Evaluate

$$\lim_{n\to\infty} \left(\frac{3n+1}{3n-1}\right)^n .$$

12soln. The $\lim_{n\to\infty} \left(\frac{3n+1}{3n-1}\right)^n$ converges to $e^{2/3}$.

$$\lim_{n\to\infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim_{n\to\infty} \exp\left(n \ln\left(\frac{3n+1}{3n-1}\right)\right) = \lim_{n\to\infty} \exp\left(\frac{\ln(3n+1)-\ln(3n-1)}{\frac{1}{n}}\right) = \lim_{n\to\infty} \exp\left(\frac{\frac{3}{3n+1}-\frac{3}{3n-1}}{\left(-\frac{1}{n^2}\right)}\right)$$

$$= \lim_{n\to\infty} \exp\left(\frac{6n^2}{(3n+1)(3n-1)}\right) = \exp\left(\frac{6}{9}\right) = e^{2/3} \Rightarrow \text{converges}$$

13. The formal series

$$\sum_{n=2}^{\infty} \frac{1}{n\ (\ln n)^2} .$$

13soln. The series $\sum \frac{1}{n\ (\ln n)^2}$ converges, as can be shown by the integral test.

The function $f(x) = \dfrac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t \quad \text{[by substitution with } u = \ln x] \quad = -\lim_{t\to\infty}\left(\frac{1}{\ln t} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2},$$

so the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

14. Find **all** real numbers r satisfying that

$$\sum_{n=2}^{\infty} r^n = \frac{1}{20} .$$

14soln. Soln: $-\frac{1}{4}$ and $\frac{1}{5}$

First note that for the series $\sum_{n=2}^{\infty} r^n$ to converge (so that the problem even makes sense), we need

$$|r| < 1.$$

So let $\boxed{|r| < 1}$. Next, to find the sum $\sum_{n=2}^{\infty} r^n$, consider the partial sums

$$s_n \overset{\text{def}}{=} r^2 + r^3 + \ldots + r^{n-1} + r^n \overset{\text{i.e.}}{=} \sum_{k=2}^{n} r^k.$$

Cancellation Heaven occurs with a geometric series when one computes $s_n - rs_n$. Let's see why.

$$\begin{aligned} s_n &= r^2 + r^3 + \ldots + r^{n-1} + r^n \\ r\,s_n &= \phantom{r^2 +{}} r^3 + r^4 + \ldots + r^n + r^{n+1} \end{aligned}$$

Do you see the cancellation that would occur if we take $s_n - rs_n$?

$$\begin{array}{rcl} s_n & = & r^2 + \cancel{r^3} + \ldots + \cancel{r^{n-1}} + \cancel{r^n} \\ & & \swarrow \quad \cdot\swarrow\cdot \quad \swarrow \\ r\,s_n & = & \cancel{r^3} + \cancel{r^4} + \ldots + \cancel{r^n} + r^{n+1} \end{array}$$

substract ——————————————————————

$$(1-r)\, s_n \overset{\text{Ⓐ}}{=} s_n - rs_n = r^2 - r^{n+1}$$

So, since $r \neq 1$,

$$s_n = \frac{r^2 - r^{n+1}}{1-r}\,.$$

Since $|r| < 1$, we have that $\lim_{n\to\infty} r^n = 0$. So

$$\sum_{k=2}^{n} r^k \overset{\text{def}}{=} s_n = \frac{r^2 - r^{n+1}}{1-r} \xrightarrow{n\to\infty} \frac{r^2}{1-r} = \sum_{n=2}^{\infty} r^n.$$

in other words,

$$\sum_{n=2}^{\infty} r^n \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=2}^{n} r^k \overset{\text{i.e.}}{=} \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{r^2 - r^{n+1}}{1-r} = \frac{r^2}{1-r}\,.$$

So we are looking for $r \in \mathbb{R}$ so that $|r| < 1$ and

$$\frac{r^2}{1-r} = \frac{1}{20}.$$

Note $\left[\frac{r^2}{1-r} = \frac{1}{20}\right] \Leftrightarrow [20r^2 = 1 - r] \Leftrightarrow [20r^2 + r - 1 = 0] \Leftrightarrow$

$$r = \frac{-1 \pm \sqrt{1 + 4\,(20)}}{2\,(20)} = \frac{-1 \pm 9}{2\,(20)} = \begin{cases} \frac{-1+9}{2(20)} = \frac{1}{5} \\ \frac{-1+9}{2(20)} = -\frac{1}{4}. \end{cases}$$