MARK BOX		
PROBLEM	POINTS	
0	30	
1-10	50=10x5	
11	10	
12	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **MultipleChoice problems 1–10**, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
- **For problems > 10**, to receive credit you **MUST**:
  (1) **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  (2) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (3) if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1-8.5, 8.7, 8.8 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

0. Fill-in the blanks.

0.1. $\arccos(\,\text{-}\frac{1}{2}\,) = \underline{\quad \frac{2\pi}{3} \quad}$ (Your answers should be an angle in **RADIANS**.)

0.2. $\arcsin(\,\text{-}\frac{1}{2}\,) = \underline{\quad \frac{-\pi}{6} \quad}$ (Your answers should be an angle in **RADIANS**.)

0.3/0.4. Double-angle Formulas. Your answer should involve trig functions of θ, and not of 2θ.

$\cos(2\theta) = \underline{\quad \cos^2\theta \;-\; \sin^2\theta \quad}$ and $\sin(2\theta) = \underline{\quad 2\,\sin\theta\cos\theta \quad}$

0.5/0.6. Half-Angle Formula. Your answer should involve $\cos(2\theta)$.

$\cos^2(\theta) = \underline{\quad \dfrac{1+\cos(2\theta)}{2} \quad}$ and $\sin^2(\theta) = \underline{\quad \dfrac{1-\cos(2\theta)}{2} \quad}$.

0.7. $\int \frac{du}{u} \overset{u\neq 0}{=} \underline{\quad \ln\,|u| \quad} + \text{C}$

0.8. $\int u^n\,du \overset{n\neq -1}{=} \underline{\quad \frac{u^{n+1}}{n+1} \quad} + \text{C}$

0.9. $\int e^u\,du = \underline{\quad e^u \quad} + \text{C}$

0.10. $\int \cos u\,du = \underline{\quad \sin u \quad} + C$

0.11. $\int \sec^2 u\,du = \underline{\quad \tan u \quad} + C$

0.12. $\int \sec u \tan u\,du = \underline{\quad \sec u \quad} + C$

0.13. $\int \sin u\,du = \underline{\quad \text{-}\cos u \quad} + C$

0.14. $\int \csc^2 u\,du = \underline{\quad \text{-}\cot u \quad} + C$

0.15. $\int \csc u \cot u\,du = \underline{\quad \text{-}\csc u \quad} + C$

0.16. $\int \tan u\,du = \underline{\quad \ln|\sec u| \overset{or}{=} \text{-}\ln|\cos u| \quad} + C$

0.17. $\int \cot u\,du = \underline{\quad \text{-}\ln|\csc u| \overset{or}{=} \ln|\sin u| \quad} + C$

0.18. $\int \sec u\,du = \underline{\quad \ln|\sec u + \tan u| \overset{or}{=} \text{-}\ln|\sec u - \tan u| \quad} + C$

0.19. $\int \csc u\,du = \underline{\quad \text{-}\ln|\csc u + \cot u| \overset{or}{=} \ln|\csc u - \cot u| \quad} + C$

0.20. $\int \frac{1}{\sqrt{a^2-u^2}}\, du \overset{a>0}{=}$ $\sin^{-1}\left(\frac{u}{a}\right)$ $+C$

0.21. $\int \frac{1}{a^2+u^2}\, du \overset{a>0}{=}$ $\frac{1}{a}\ \tan^{-1}\left(\frac{u}{a}\right)$ $+C$

0.22. $\int \frac{1}{u\sqrt{u^2-a^2}}\, du \overset{a>0}{=}$ $\frac{1}{a}\ \sec^{-1}\left(\frac{|u|}{a}\right)$ $+C$

0.23. Trig sub.: if the integrand involves $a^2 - u^2$, then one makes the substitution $u\ =$ $a \sin\theta$

0.24. Trig sub.: if the integrand involves $u^2 - a^2$, then one makes the substitution $u\ =$ $a \sec\theta$

0.25. Integration by parts formula: $\int u\, dv\ =$ $uv - \int v du$

Improper Integrals

Below, $a, b, c \in \mathbb{R}$ with $a < c < b$.

0.26. If $f\colon [0, \infty) \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_0^\infty f(x)\ dx$ by

$$\int_0^\infty f(x)\ dx = \boxed{\lim_{t\to\infty} \int_0^t f(x)\ dx}\ .$$

0.27. If $f\colon (-\infty, \infty) \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_{-\infty}^\infty f(x)\ dx$ by

$$\int_{-\infty}^\infty f(x)\ dx = \boxed{\left[\lim_{t\to-\infty} \int_t^0 f(x)\ dx\right] + \left[\lim_{s\to\infty} \int_0^s f(x)\ dx\right]}\ .$$

0.28. If $f\colon [a, c) \cup (c, b] \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_a^b f(x)\ dx$ by

$$\int_a^b f(x)\ dx = \boxed{\left[\lim_{t\to c^-} \int_a^t f(x)\ dx\right] + \left[\lim_{s\to c^+} \int_s^b f(x)\ dx\right]}\ .$$

0.29. An improper integral as above *converges* precisely when

each of the limits involves converges to a **finite** number.

0.30. An improper integral as above *diverges* precisely when

the improper integral does not converge.

MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to the multiple choice problems.
- You may choice up to **2** answers for each multiple choice problem. The scoring is as follows.
  - ∗ For a problem with precisely one answer marked and the answer is correct, 5 points.
  - ∗ For a problem with precisely two answers marked, one of which is correct, 2 points.
  - ∗ For a problem with nothing marked (i.e., left blank) 1 point.
  - ∗ All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					

5	2	1	0
Extra Credit:			

11. Show your work below the box then put your answer in the box. Your final answer should not have a trig function in it.

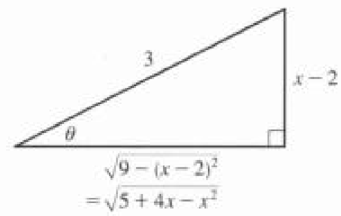
11a. Complete the square. $x^2 - 4x - 5 =$ $\boxed{(x-2)^2 - 9}$

$\left(x - \frac{4}{2}\right)^2 = (x-2)^2 = x - 4x + 4$
$x^2 - 4x - 5 = (x - 4x + 4 - 4) - 5 = (x - 4x + 4) + (-4 - 5) = (x-2)^2 - 9$

11b. $$\boxed{\int \sqrt{5 + 4x - x^2}\, dx = \frac{9}{2} \arcsin\left(\frac{x-2}{3}\right) + \frac{(x-2)\sqrt{5+4x-x^2}}{2} + \text{C}}$$

$5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x-2)^2 + 9$. Let
$x - 2 = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\, d\theta$. Then

$$\begin{aligned}
\int \sqrt{5+4x-x^2}\, dx &= \int \sqrt{9-(x-2)^2}\, dx = \int \sqrt{9 - 9\sin^2\theta}\, 3\cos\theta\, d\theta \\
&= \int \sqrt{9\cos^2\theta}\, 3\cos\theta\, d\theta = \int 9\cos^2\theta\, d\theta \\
&= \tfrac{9}{2}\int (1 + \cos 2\theta)\, d\theta = \tfrac{9}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C \\
&= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + C = \tfrac{9}{2}\theta + \tfrac{9}{4}(2\sin\theta\,\cos\theta) + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3} + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C
\end{aligned}$$



12. Derive a reduction formula for $\int x^n e^x\, dx$ where $n \in \mathbb{N} = \{1, 2, 3, 4, \ldots\}$.
Show your work below the box then put your answer in the box.

$$\boxed{\int x^n e^x\, dx = x^n e^x - n\int x^{n-1} e^x\, dx}$$

Integration by Parts' Key Idea:
► For $\int x^n f(x)\, dx$ where $\int f(x)\, dx$ is easy, try $u = x^n$ and $dv = f(x)\, dx$.
(Note that then $v = \int dv = \int f(x)\, dx$.) This often reduces x^n to x^{n-1}.

$u = x^n$	$dv = e^x\, dx$
$du = nx^{n-1}\, dx$	$v = e^x$

So Integration by Parts $\int u\, dv = uv - \int v\, du$ gives us that $\int x^n e^x\, dx = x^n e^x - n\int x^{n-1} e^x\, dx$.

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then: $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r\ln a$.

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1. Evaluate the integral

$$\int_0^1 \frac{x}{x^2+9}\,dx.$$

1soln.

$$\int_{x=0}^{x=1} \frac{x}{x^2+9}\,dx = \frac{1}{2}\int_{u=9}^{u=10} \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_{u=9}^{u=10}$$

$$= \frac{1}{2}\ln 10 - \frac{1}{2}\ln 9$$

$$\boxed{\begin{array}{l} u = x^2+9 \\ du = 2x\,dx \\ \hline x=0 \Rightarrow u=9 \\ x=1 \Rightarrow u=10 \end{array}}$$

2. Evaluate the integral

$$\int_0^4 \frac{x}{x+9}\,dx\ .$$

2soln.

$$\int_0^4 \frac{x}{x+9}\,dx$$

Do not have strictly bigger bottoms so need to do long division.
But it's easy to "fake" long division here:

$$\frac{x}{x+9} = \frac{x+9-9}{x+9} = \frac{x+9}{x+9} - \frac{9}{x+9} = 1 - \frac{9}{x+9}.$$

So

$$\int_0^4 \frac{x}{x+9}\,dx = \int_0^4 \left[1 - \frac{9}{x+9}\right] dx$$

$$= \left[x - 9\ln|x+9|\ \right]\Big|_{x=0}^{x=4}$$

$$= (4 - 9\ln 13) - (0 - 9\ln 9)$$

$$= 4 - 9\ln 13 + 9\ln 9.$$

3. Evaluate

$$\int_0^{\ln(2\pi)} e^x \cos\left(e^x\right)\,dx$$

3soln. Let $u = e^x$. So $du = e^x dx$. So $\int e^x \cos\left(e^x\right)\ dx = \int \cos u\, du = \sin u + C = \sin\left(e^x\right) + C$.
Next check indefinite integral: $D_x \sin\left(e^x\right) = \left[\cos\left(e^x\right)\right] D_x e^x = \left[\cos\left(e^x\right)\right] e^x$ $\boxed{\checkmark}$.
So $\int_0^{\ln(2\pi)} e^x \cos\left(e^x\right)\ dx = \sin e^x \Big|_{x=0}^{x=\ln(2\pi)} = \sin e^{\ln(2\pi)} - \sin e^0 = \sin(2\pi) - \sin 1 = 0 - \sin 1 = -\sin 1$

4. Evaluate

$$\int_{x=0}^{x=\frac{3\pi}{2}} e^x \cos x \, dx \, .$$

4soln. Below we show that

$$\int e^x \cos x \, dx = \frac{e^x(\sin x + \cos x)}{2} + C.$$

So

$$\int_{x=0}^{x=\frac{3\pi}{2}} e^x \cos x \, dx = \frac{e^x(\sin x + \cos x)}{2} \Big|_0^{3\pi/2} = \frac{e^{3\pi/2}(-1)}{2} - \frac{e^0(1)}{2} = \frac{-1 - e^{3\pi/2}}{2}.$$

To find the indefinite integral, use two integration by parts and the *bring to the other side* idea. For the two integration by parts, put the expontential function with either the u's both times or the dv's both times.

$\boxed{\text{Way \# 1}}$

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{1x} \qquad dv_1 = \cos 1x \, dx$$

$$du_1 = 1e^{1x} \, dx \qquad v_1 = \frac{1}{1} \sin 1x \, .$$

So by integration by parts

$$\int e^{1x} \cos 1x \, dx \;=\; \frac{1}{1} e^{1x} \sin 1x - \frac{1}{1} \int e^{1x} \sin 1x \, dx \, .$$

Now let

$$u_2 = e^{1x} \qquad dv_2 = \sin 1x \, dx$$

$$du_2 = 1e^{1x} \, dx \qquad v_2 = \frac{-1}{1} \cos 1x \, .$$

to get

$$\begin{aligned} \int e^{1x} \cos 1x \, dx \;&=\; \frac{1}{1} e^{1x} \sin 1x - \frac{1}{1} \left[\frac{-1}{1} e^{1x} \cos 1x - \frac{-1}{1} \int e^{1x} \cos 1x \, dx \right] \\ &=\; \frac{1}{1} e^{1x} \sin 1x + \frac{1}{1^2} e^{1x} \cos 1x - \frac{1^2}{1^2} \int e^{1x} \cos 1x \, dx \, . \end{aligned}$$

Now solving for $\int e^{1x} \cos 1x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{1^2}{1^2} \right] \int e^{1x} \cos 1x \, dx \;=\; \frac{1}{1} e^{1x} \sin 1x + \frac{1}{1^2} e^{1x} \cos 1x + K$$

and so

$$\begin{aligned} \int e^{1x} \cos 1x \, dx \;&=\; \left[\frac{1^2}{2} \right] \left(\frac{1}{1} e^{1x} \sin 1x + \frac{1}{1^2} e^{1x} \cos 1x + K \right) \\ &=\; \frac{1}{2} \, e^{1x} \sin 1x + \frac{1}{2} \, e^{1x} \cos 1x + \left[\frac{K1^2}{2} \right] \\ &=\; \frac{e^{1x}}{2} \, (1 \sin 1x + 1 \cos 1x) + \left[\frac{K1^2}{2} \right] \, . \end{aligned}$$

Thus

$$\int e^{1x} \cos 1x \, dx = \boxed{\frac{e^{1x}}{2} \, (1 \cos 1x + 1 \sin 1x) + C} .$$

Way # 2

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$u_1 = \cos 1x \qquad dv_1 = e^{1x} dx$$

$$du_1 = -1 \sin 1x \, dx \qquad v_1 = \frac{1}{1} e^{1x} .$$

So, by integration by parts

$$\int e^{1x} \cos 1x \, dx = \frac{1}{1} e^{1x} \cos 1x - \frac{-1}{1} \int e^{1x} \sin 1x \, dx .$$

Now let

$$u_2 = \sin 1x \qquad dv_2 = e^{1x} dx$$

$$du_2 = 1 \cos 1x \, dx \qquad v_2 = \frac{1}{1} e^{1x} .$$

to get

$$\begin{aligned} \int e^{1x} \cos 1x \, dx &= \frac{1}{1} e^{1x} \cos 1x + \frac{1}{1} \left[\frac{1}{1} e^{1x} \sin 1x - \frac{1}{1} \int e^{1x} \cos 1x \, dx \right] \\ &= \frac{1}{1} e^{1x} \cos 1x + \frac{1}{1^2} e^{1x} \sin 1x - \frac{1^2}{1^2} \int e^{1x} \cos 1x \, dx . \end{aligned}$$

Now solving for $\int e^{1x} \cos 1x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{1^2}{1^2} \right] \int e^{1x} \cos 1x \, dx = \frac{1}{1} e^{1x} \cos 1x + \frac{1}{1^2} e^{1x} \sin 1x + K$$

and so

$$\begin{aligned} \int e^{1x} \cos 1x \, dx &= \left[\frac{1^2}{1^2 + 1^2} \right] \left(\frac{1}{1} e^{1x} \cos 1x + \frac{1}{1^2} e^{1x} \sin 1x + K \right) \\ &= \frac{1}{2} \, e^{1x} \cos 1x + \frac{1}{2} \, e^{1x} \sin 1x + \left[\frac{K 1^2}{1^2 + 1^2} \right] \\ &= \frac{e^{1x}}{2} \, (1 \cos 1x + 1 \sin 1x) + \left[\frac{K 1^2}{1^2 + 1^2} \right] \end{aligned}$$

Thus

$$\int e^{1x} \cos 1x \, dx = \boxed{\frac{e^{1x}}{2} \, (1 \cos 1x + 1 \sin 1x) + C} .$$

Doesn't Work Way

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

5. Investigate the convergence of
$$\int_{x=1}^{x=\infty} \frac{1-e^{-x}}{x}\, dx \ .$$

5soln. This is Example 9 from § 8.8 of our book by Thomas.

TABLE 8.5

b	$\int_1^b \frac{1-e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$.

Solution The integrand suggests a comparison of $f(x) = (1-e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \leq g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1-e^{-x}) = 1,$$
which is a positive finite limit. Therefore, $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$. ■

6. Evaluate
$$\int_{x=0}^{x=1} \sin^4 x\, dx \ .$$

6soln. From Class Handout on Trig. Substitution: **Example 4.** $\int \sin^4 x\, dx$.
u-du sub does not work (why? e.g.: $\int \sin^4 x\, dx \overset{u=\cos x}{=} -\int \boxed{\sin^3 x}\,[-\sin x\, dx]$).
For $\int \sin^n x \cos^m x\, dx$, with BOTH $m, n \in \{0, 2, 4, 6, \ldots\}$, use the half-angle formulas.

$$\int \sin^4 x\, dx \overset{\text{alg.}}{=} \int [\sin^2 x]^2\, dx \overset{(\frac{1}{2}\angle)}{=} \int \left[\frac{1-\cos(2x)}{2}\right]^2 dx \overset{\text{alg.}}{=} \frac{1}{4}\int [1 - 2\cos(2x) + \cos^2(2x)]\, dx$$
$$\overset{(\frac{1}{2}\angle)}{=} \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] dx$$
$$= \frac{1}{4}\int dx - \frac{1}{4}\int 2\cos(2x)\, dx + \frac{1}{4}\cdot\frac{1}{2}\int dx + \frac{1}{4}\cdot\frac{1}{2}\int \cos(4x)\, dx$$
$$= \frac{1}{4}\left(1+\frac{1}{2}\right)\int dx - \frac{1}{4}\cdot\int \cos(2x)\ [2dx] + \left(\frac{1}{4}\cdot\frac{1}{2}\right)\cdot\left(\frac{1}{4}\right)\int \cos(4x)\ [4dx]$$
$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$

So, since $\sin 0 = 0$,
$$\int_{x=0}^{x=1} \sin^4 x\, dx = \left[\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right]\Big|_{x=0}^{x=1}$$
$$= \left[\frac{3}{8} - \frac{1}{4}\sin 2 + \frac{1}{32}\sin 4\right] - [0 - 0 + 0]$$
$$= \frac{3}{8} - \frac{1}{4}\sin 2 + \frac{1}{32}\sin 4.$$

7. Evaluate

$$\int_{x=5}^{x=10} \frac{\sqrt{x^2-25}}{x}\,dx$$

AND specify the initial substitution.

7soln. The integrand has a $u^2 - a^2$, so we let $u = a\sec\theta$. $\boxed{x = 5sec\theta \quad \text{so} \quad dx = 5\sec\theta\tan\theta\,d\theta}$

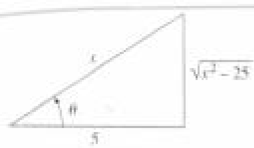
Thus,

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= 5\int \tan^2\theta\,d\theta \qquad \tan\theta \ge 0 \text{ since } 0 \le \theta < \pi/2$$

$$= 5\int (\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C$$



$x = 5\sec\theta$

Figure 8.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 8.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

• Check $D_x\left[(x^2-25)^{1/2} - 5\sec^{-1}\left(\frac{x}{5}\right)\right]$

$$= \frac{1}{2}(x^2-25)^{-1/2}(2x) - 5\,\frac{1}{\left|\frac{x}{5}\right|\sqrt{\left(\frac{x}{5}\right)^2 - 1}} \cdot \frac{1}{5}$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{1}{\frac{x}{5}\sqrt{\frac{x^2}{25} - \frac{25}{25}}} \qquad (\text{know } x \ge 5)$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{25}{x(x^2-25)^{1/2}} = \frac{(x^2-25)}{x(x^2-25)^{1/2}} \cdot \frac{(x^2-25)^{1/2}}{(x^2-25)^{1/2}}$$

$$= \frac{(x^2-25)(x^2-25)^{1/2}}{x(x^2-25)} = \frac{\sqrt{x^2-25}}{x}. \checkmark$$

• $$\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)\Big|_{x=5}^{x=10}$$

$$= \left[\sqrt{100-25} - 5\sec^{-1}2\right] - \left[0 - 5\cancel{\sec^{-1}1}^{\,0}\right] =$$

$$= \sqrt{75} - 5\cdot\frac{\pi}{3} = 5\sqrt{3} - 5\left(\frac{\pi}{3}\right) = 5\left(\sqrt{3} - \frac{\pi}{3}\right).$$

8. Evaluate

$$\int_{x=1}^{x=3} \frac{5x^2+3x-2}{x^3+2x^2}\,dx\ .$$

8soln.

- $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right) dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

- Check $D_x\left[2\ln|x| + x^{-1} + 3\ln|x+2|\right] = \frac{2}{x} + -1x^{-2} + \frac{3}{x+2}$

$$= \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2} = \frac{2x(x+2) - (x+2) + 3x^2}{x^2(x+2)} = \frac{5x^2+3x-2}{x^3+2x^2} \checkmark$$

- $\left[3\ln|x+2| + 2\ln x + \frac{1}{x}\right]\Big|_{x=1}^{x=3} =$

$$\left[3\ln 5 + 2\ln 3 + \tfrac{1}{3}\right] - \left[3\ln 3 + \underbrace{2\ln 1}_{=0} + 1\right] =$$

$$3\ln 5 - \ln 3 - \tfrac{2}{3}\ .$$

9. For which value of p does

$$\int_0^1 \frac{1}{x^p}\,dx = 1.25\ ?$$

9soln. First compute

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0+}\int_t^1 x^{-p}\,dx \overset{p\neq 1}{=} \lim_{t\to 0+}\frac{x^{1-p}}{1-p}\Big|_t^1 = \frac{1}{1-p}\lim_{t\to 0+} x^{1-p}\big|_t^1 = \frac{1}{1-p}\lim_{t\to 0+}\left[1 - t^{1-p}\right].$$

If $1-p>0$, or equivalently $1>p$, then $\lim_{t\to 0+} t^{1-p} = 0$ and so

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{t\to 0+}\left[1-t^{1-p}\right] = \frac{1}{1-p}\left[1-0\right] = \frac{1}{1-p}.$$

If $1-p<0$, or equivalently $1<p$, then $\lim_{t\to 0+} t^{1-p} = \infty$ and so

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{t\to 0+}\left[1-t^{1-p}\right] = \infty.$$

If $p=1$, then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0+}\ln x\big|_t^1 = \lim_{t\to 0+}\left[0 - \ln t\right] = \infty.$$

So we need

$$p<1 \quad\text{and}\quad \frac{1}{1-p} = 1.25.$$

Note

$$\frac{1}{1-p} = 1.25 = \frac{5}{4} = \frac{1}{\frac{4}{5}} \quad\Leftrightarrow\quad 1-p = \frac{4}{5} \quad\Leftrightarrow\quad p = 1 - \frac{4}{5} = \frac{1}{5} = 0.2.$$

10. Evaluate the integral

$$\int_{x=-1}^{x=1} \frac{1}{x^3}\, dx \ .$$

10soln.

• $\int x^{-3}\, dx = \frac{x^{-2}}{-2} + C$

$\int_{x=0}^{x=1} x^{-3}\, dx = \lim_{a \to 0^+} \left. \frac{x^{-2}}{-2} \right|_{x=a}^{x=1} = \frac{1}{2} \lim_{a \to 0^+} \left[\frac{1}{x^2} \right]_{x=1}^{x=a} =$

$\frac{1}{2} \lim_{x \to 0^+} \left[\frac{1}{a^2} - 1 \right] = \infty$. Similarly, $\int_{-1}^{0} x^{-3}\, dx = -\infty$

• $\int_{-1}^{1} x^{-3}\, dx = \int_{-1}^{0} x^{-3}\, dx + \int_{0}^{1} x^{-3}\, dx = -\infty + \infty$ so DNE.