1. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) \;=\; \frac{2}{3-x}$$

about the center $x_0 = 0$ and state when this Taylor series is valid. Hint, by simple algebra,

$$f(x) \;=\; \frac{2}{3-x} \;=\; \left(\frac{2}{3}\right)\left(\frac{1}{1-\frac{x}{3}}\right).$$

a. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n x^n$, valid for $|x| < 1$

b. $\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n$, valid for $|x| < 3$

c. $\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n$, valid for $|x| < 1$

d. $\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n$, valid for $|x| < 3$

e. none of these

2. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) \;=\; \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$. Hint. Start with the Taylor series expansion

$$\frac{1}{1-x} \;=\; \sum_{k=0}^{\infty} x^k \qquad \text{valid for } |x| < 1$$

and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} \;=\; (-1)(1-x)^{-2}\,D_x(1-x) \;=\; (-1)(1-x)^{-2}\,(-1) \;=\; (1-x)^{-2}\,.$$

a. $\sum_{n=0}^{\infty} \frac{(n)(n-1)(n-2)}{6}\, x^{n-3}$

b. $\sum_{n=0}^{\infty} (n)(n-1)(n-2)\, x^n$

c. $\sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6}\, x^n$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{(n+3)(n+2)(n+1)}{6}\, x^n$

e. none of these

3. Using a known (commonly used) Taylor series, evaluate $\int \tan^{-1}(t^2)\,dt$ as a power series.

a. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+3}}{(4n+3)}$

b. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+3}}{(2n+1)(4n+3)}$

c. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+2}}{(2n+3)}$

d. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+2}}{(2n+1)}$

e. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+3}}{(2n+3)}$

4. Find the 3rd order Taylor polynomial for $f(x) = \frac{1}{x}$ about the center $x_0 = 2$.

a. $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

b. $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}(x-2)^2 - \frac{3}{8}(x-2)^3$

c. $\frac{1}{2} + \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 + \frac{1}{16}(x-2)^3$

d. $\frac{1}{2} - \frac{1}{4}x + \frac{1}{4}x^2 - \frac{3}{8}x^3$

e. none of these

5. Find the Taylor series for $f(x) = x^4 - 3x^2 + 1$ about the center $x_0 = 1$.

a. $(x-1)^4 - 3(x-1)^2 + 1$

b. $-1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$

c. $-1 - 2(x-1) + 6(x-1)^2 + 24(x-1)^3 + 24(x-1)^4$

d. $-1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4 + 2(x-1)^5$

e. none of these

6. Find the Taylor series for

$$f(x) = \frac{1}{x^2}$$

about the center $x_0 = 1$.

a. $\sum_{n=0}^{\infty}(-1)^n\,(n+1)!\,x^n$ b. $\sum_{n=0}^{\infty}(-1)^n\,(n+1)!\,(x-1)^n$ c. $\sum_{n=0}^{\infty}(-1)^n\,(n+1)\,(x-1)^n$

d. $\sum_{n=0}^{\infty}(-1)^{n+1}\,(n+1)\,(x-1)^n$ e. none of these

7. Consider the function

$$f(x) = e^{-x}\ .$$

The 5th order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_5(x) \;=\; \sum_{n=0}^{5}\frac{(-x)^n}{n!} \;=\; 1 \;-\; x \;+\; \frac{x^2}{2!} \;-\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;-\; \frac{x^5}{5!}\ .$$

The 5th order Remainder term $R_5(x)$ is defined by $R_5(x) = f(x) - P_5(x)$ and so $e^{-x} \approx P_5(x)$ where the approximation is within an error of $|R_5(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_5(x)|$ that is valid for each $x \in (7, 9)$.

a. $\frac{(e^{9})(9^5)}{5!}$ b. $\frac{(e^{-9})(9^6)}{6!}$ c. $\frac{(e^{-7})(9^6)}{6!}$ d. $\frac{(e^{-0})(9^6)}{6!}$ e. none of these

8. In this problem, you must show your work. Clearly explain your thought process. Using Taylor's (BIG) Remainder Theorem, show that

$$\sum_{n=0}^{\infty}\frac{1}{n!} \;=\; e\ .$$

Hint: The instruction says to use Taylor's Remainder Theorem so you cannot use the facts listed on the *Commonly Used Taylor Series* handout. (Indeed, the facts listed on the *Commonly Used Taylor Series* handout are shown by using Taylor's Remainder Theorem so think of this problem as showing one of these facts.)

Hint. Consider the function $f(x) = e^x$.