HAND IN PART

MARK BOX		
PROBLEM	POINTS	
1-25	100	
%	100	

NAME: Work Moose

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a students request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1–8.5, 8.7–8.8, 10.1–10.10, 11.1–11.5 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________________

⋇ Indicate (by circling) directly in the table below your solution to each problem.
⋇ You may choice up to **2** answers for each problem. The scoring is as follows.

- For a problem with precisely one answer marked and the answer is correct, 4 points.
- For a problem with precisely two answers marked, one of which is correct, 2 points.
- For a problem with nothing marked (i.e., left blank), 1 point.
- All other cases, 0 points.

⋇ Fill in the NUMBER OF SOLUTIONS CIRCLED column.

Your Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	4	2	1	0
1	1a	1b	1c	**1d**	1e					
2	**2a**	2b	2c	2d	2e					
3	3a	**3b**	3c	3d	3e					
4	4a	4b	4c	**4d**	4e					
5	5a	5b	**5c**	5d	5e					
6	6a	**6b**	6c	6d	6e					
7	7a	7b	7c	**7d**	7e					
8	**8a**	8b	8c	8d	8e					
9	9a	9b	**9c**	9d	9e					
10	10a	10b	10c	**10d**	10e					
11	**11a**	11b	11c	11d	11e					
12	12a	12b	**12c**	12d	12e					
13	13a	13b	13c	**13d**	13e					
14	14a	14b	**14c**	14d	14e					
15	15a	**15b**	15c	15d	15e					
16	16a	16b	**16c**	16d	16e					
17	17a	**17b**	17c	17d	17e					
18	18a	**18b**	18c	18d	18e					
19	19a	**19b**	19c	19d	19e					
20	20a	20b	20c	**20d**	20e					
21	21a	21b	**21c**	21d	21e					
22	22a	**22b**	22c	22d	22e					
23	23a	23b	23c	**23d**	23e					
24	24a	24b	**24c**	24d	24e					
25	**25a**	**25b**	**25c**	**25d**	**25e**	17				

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

- Hint. For a typical (i.e. not improper) definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.

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1. Evaluate the integral

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x \; dx \; .$$

1soln. See hints above. Since

$$\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx \overset{u=\sin x}{=} \int (1-u^2)\,du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

we have that

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x\,dx = \left(\sin x - \frac{1}{3}\sin^3 x\right) \Big|_{x=0}^{x=\frac{\pi}{2}} = \left(1 - \frac{1}{3}\right) - (0-0) = \frac{2}{3} \; .$$

2. Evaluate the integral

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^2 x \; dx \; .$$

2soln. See hints above for why we do indefinite integral first. Since

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x \cdot 2dx$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

(and next you should take $D_x(\frac{x}{2} + \frac{\sin 2x}{4})$ and if you do indeed get $\cos^2 x$ then you can proceed to next step of) we have that

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^2 x \; dx \; = \left(\frac{x}{2} + \frac{1}{4}\sin 2x\right) \Big|_{x=0}^{x=\frac{\pi}{2}} = \left(\frac{\pi}{4} + 0\right) + (0+0) = \frac{\pi}{4}$$

3. Evaluate the integral

$$\int_{x=1}^{x=e} \ln x \; dx \; .$$

3soln. Since

EXAMPLE 2 Evaluate $\int \ln x \, dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then $$du = \frac{1}{x}\,dx \qquad v = x$$

Integrating by parts, we get

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \frac{dx}{x} \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f.

we have that

$$\int_{x=1}^{x=e} \ln x \; dx \;= (x\ln x - x)\,|_{x=1}^{x=e} = (e\ln e - e) - (1\ln 1 - 1) = (e-e) - (0-1) = 1$$

4. Evaluate the integral

$$\int_{x=0}^{x=\pi} e^{3x} \; \cos 2x \; dx \; .$$

4soln. Two integration by parts and the *bring to the other side* idea. For both the integration by parts, either let both u's involve the expontential function or let both dv's involve the expontential function; we'll do the latter way. So let

$$\begin{aligned} u_1 &= \cos 2x & dv_1 &= e^{3x} dx \\ du_1 &= -2 \sin 2x \, dx & v_1 &= \frac{1}{3}\, e^{3x} \; . \end{aligned}$$

So, by integration by parts

$$\int e^{3x} \; \cos 2x \; dx \;=\; \frac{1}{3} e^{3x} \cos 2x - \frac{-2}{3} \int e^{3x} \sin 2x \, dx \; .$$

Now let

$$u_2 = \sin 2x \qquad dv_2 = e^{3x} dx$$
$$du_2 = 2\cos 2x \, dx \qquad v_2 = \frac{1}{3} e^{3x} \; .$$

to get

$$\begin{aligned} \int e^{3x} \cos 2x \; dx &= \frac{1}{3} e^{3x} \cos 2x + \frac{2}{3}\left[\frac{1}{3} e^{3x} \sin 2x - \frac{2}{3}\int e^{3x}\cos 2x \, dx\right] \\ &= \frac{1}{3} e^{3x} \cos 2x + \frac{2}{3^2} e^{3x} \sin 2x - \frac{2^2}{3^2}\int e^{3x}\cos 2x \, dx \; . \end{aligned}$$

Now to solve for $\int e^{3x} \cos 2x \, dx$ (use the *bring to the other side* idea) we have

$$\left[1 + \frac{2^2}{3^2}\right]\int e^{3x} \cos 2x \; dx = \frac{1}{3} e^{3x} \cos 2x + \frac{2}{3^2} e^{3x} \sin 2x + C$$

and so

$$\begin{aligned} \int e^{3x} \cos 2x \; dx &= \left[\frac{3^2}{3^2 + 2^2}\right]\left(\frac{1}{3} e^{3x} \cos 2x + \frac{2}{3^2} e^{3x} \sin 2x + C\right) \\ &= \frac{3}{13}\, e^{3x} \cos 2x + \frac{2}{13}\, e^{3x} \sin 2x + \left[\frac{C3^2}{3^2 + 2^2}\right] \\ &= \frac{e^{3x}}{13}\,(3\cos 2x + 2\sin 2x) + \left[\frac{C3^2}{3^2 + 2^2}\right] \end{aligned}$$

Thus

$$\begin{aligned} \int_{x=0}^{x=\pi} e^{3x} \cos 2x \; dx &= \frac{e^{3x}}{13}\,(3\cos 2x + 2\sin 2x)\,\Big|_{x=0}^{x=\pi} = \left[\frac{e^{3\pi}}{13}\,(3+0)\right] - \left[\frac{e^{0}}{13}\,(3+0)\right] \\ &= \frac{3}{13}\left(e^{3\pi} - 1\right) \; . \end{aligned}$$

5. Evaluate the integral

$$\int_{x=0}^{x=1} \frac{1}{\sqrt{4 + x^2}} \, dx \; .$$

Do not overlook the square root sign in the denominator.

5soln. For the indefinte integral, let

We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta \, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$
$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

URE 8.4 Reference triangle for = 2 tan θ (Example 1):

$\tan\theta = \frac{x}{2}$

$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$

Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C. \qquad \text{From Fig. 8.4}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle.

So

$$\int_{x=0}^{x=1} \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left|\frac{\sqrt{4+1}}{2}+\frac{1}{2}\right| - \ln\left|\frac{\sqrt{4+0}}{2}+\frac{0}{2}\right| = \ln\left|\frac{\sqrt{5}}{2}+\frac{1}{2}\right| - \ln\left|1+\frac{0}{2}\right| = \ln\left|\frac{\sqrt{5}}{2}+\frac{1}{2}\right| .$$

6. Evaluate the integral

$$\int_{x=0}^{x=\frac{\sqrt{3}}{2}} \frac{4x^2}{(1-x^2)^{3/2}}\,dx$$

AND specify the initial substitution.

6soln. Trig Subtitute with initial substitution $x = \sin\theta$. Let

$$x = \sin\theta,\ 0 \le \theta \le \tfrac{\pi}{3},\ dx = \cos\theta\, d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta;$$

$$\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{\left(1-x^2\right)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)d\theta = 4\int_0^{\pi/3}\left(\sec^2\theta - 1\right)d\theta = 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

7. Let $y = p(x)$ be a polynomial of degree 5.
What is the form of the partial fraction decomposition of

$$\frac{p(x)}{(x^2-1)\,(x^2+1)^2}\ ?$$

7soln. $(x^2-1)\,(x^2+1)^2 = (x-1)\,(x+1)\,(x^2+1)^2$ where $x-1$ and $x+1$ are linear terms while x^2+1 is an irreducible quadratic. Now see the partial fraction handout from class. Correct answer is

$$\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

8. Evaluate the integral

$$\int_{x=1}^{x=3} \frac{5x^2+3x-2}{x^3+2x^2}\,dx\ .$$

8soln.

- $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So
$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right) dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$

- Check $D_x\left[2\ln|x| + x^{-1} + 3\ln|x+2|\right] = \frac{2}{x} + -1x^{-2} + \frac{3}{x+2}$
$= \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2} = \frac{2x(x+2) - (x+2) + 3x^2}{x^2(x+2)} = \frac{5x^2+3x-2}{x^3+2x^2}$ ✓

- $\left[3\ln|x+2| + 2\ln x + \frac{1}{x}\right]\Big|_{x=1}^{x=3} =$
$\left[3\ln 5 + 2\ln 3 + \frac{1}{3}\right] - \left[3\ln 3 + \underbrace{2\ln 1}_{=0} + 1\right] =$
$3\ln 5 - \ln 3 - \frac{2}{3}$.

9. Evaluate the integral

$$\int_{x=-1}^{x=1} \frac{1}{x^6}\,dx\ .$$

9soln. Indefinite integral: $\int x^{-6}\,dx = \frac{x^{-5}}{-5} + C$.
Note that he function $y = x^{-6}$ is undefined at $x=0$; therefore, $\int_{-1}^{1} x^{-6}dx$ is an improrper integral and we need to investigate the behaviour of $\int_{-1}^{0} x^{-6}dx$ and $\int_{0}^{1} x^{-6}dx$. Note

$$\int_0^1 x^{-6}\,dx = \lim_{b\to 0^+}\int_b^1 x^{-6}\,dx = \frac{-1}{5}\lim_{b\to 0^+}\frac{1}{x^5}\Big|_{x=b}^{x=1} = \frac{-1}{5}\lim_{b\to 0^+}\left[1 - \frac{1}{b^5}\right] \cancel{\overset{(\frac{-1}{5})(-\infty)}{=}} {}^{+}\infty\ .$$

Similiarly (or can just use symmetry)

$$\int_{-1}^0 x^{-6}\,dx = \lim_{a\to 0^-}\int_{-1}^a x^{-6}\,dx = \frac{-1}{5}\lim_{a\to 0^-}\frac{1}{x^5}\Big|_{x=-1}^{x=a} = \frac{-1}{5}\lim_{a\to 0^-}\left[\frac{1}{a^5} - \frac{1}{-1}\right] \cancel{\overset{(\frac{-1}{5})(-\infty)}{=}} {}^{+}\infty\ .$$

Thus

$$\int_{-1}^1 x^{-6}\,dx = \int_{-1}^0 x^{-6}\,dx + \int_0^1 x^{-6}\,dx \overset{\text{see above}}{=} \cancel{\infty + \infty} = \infty$$

and so $\int_{-1}^1 x^{-6}\,dx$ diverges to infinity.

10. Evaluate the integral

$$\int_{x=-1}^{x=1} \frac{1}{x^5}\,dx\ .$$

10soln. Indefinite integral: $\int x^{-5}\,dx = \frac{x^{-4}}{-4} + C$.
Note that he function $y = x^{-5}$ is undefined at $x = 0$; therefore, $\int_{-1}^{1} x^{-5} dx$ is an impr511per integral and we need to investigate the behaviour of $\int_{-1}^{0} x^{-5} dx$ and $\int_{0}^{1} x^{-5} dx$. Note

$$\int_0^1 x^{-5}\,dx = \lim_{b\to 0^+} \int_b^1 x^{-5}\,dx = \frac{-1}{4} \lim_{b\to 0^+} \frac{1}{x^4} \Big|_{x=b}^{x=1} = \frac{-1}{4} \lim_{b\to 0^+} \left[1 - \frac{1}{b^4}\right] \overset{\cancel{(\frac{-1}{4})(-\infty)}}{=} {}^{+}\infty \ .$$

Similiarly (also can do by symmetry)

$$\int_{-1}^0 x^{-5}\,dx = \lim_{a\to 0^-} \int_{-1}^a x^{-5}\,dx = \frac{-1}{4} \lim_{a\to 0^-} \frac{1}{x^4} \Big|_{x=-1}^{x=a} = \frac{-1}{4} \lim_{a\to 0^-} \left[\frac{1}{a^4} - \frac{1}{-1}\right] \overset{\cancel{(\frac{-1}{4})(+\infty)}}{=} {}^{-}\infty \ .$$

So $\int_{-1}^{1} x^{-5}\,dx$ does not exist but also does not diverge to infinity.

11. Find

$$\lim_{n\to\infty} \frac{\sqrt{25n^7 - n^2 + 1}}{3n^4 + 5n^2 - 2} \ .$$

11soln. Divide through *by the dominating force* idea. Num.'s dominating force is $\sqrt{n^7} = n^{3.5}$. Den.'s dominating force is n^4. So the dominating force is $n^4 \overset{\text{i.e.}}{=} \sqrt{n^8}$. So

$$\frac{\sqrt{25n^7 - n^2 + 1}}{3n^4 + 5n^2 - 2} = \frac{\sqrt{\frac{25n^7 - n^2 + 1}{n^8}}}{\frac{3n^4 + 5n^2 - 2}{n^4}} = \frac{\sqrt{\frac{25}{n} - \frac{1}{n^6} + \frac{1}{n^8}}}{3 + \frac{5}{n^2} - \frac{2}{n^4}} \xrightarrow{n\to\infty} \frac{\sqrt{0 - 0 + 0}}{3 + 0 - 0} = 0$$

12. For how many (distinct) values of $r \in \mathbb{R}$ does

$$\sum_{n=2}^{\infty} r^n \ = \ \frac{1}{20} \ ?$$

12soln. Let $s_N = \sum_{n=2}^{N} r^n$. Then

$$s_N = r^2 + r^3 + \ldots + r^N$$
$$rs_N = r^3 + r^4 + \ldots + r^{N+1}.$$

So if $|r| < 1$, then

$$s_N = \frac{r^2 - r^{N+1}}{1 - r} \xrightarrow{N\to\infty} \frac{r^2}{1 - r} \ . \qquad (1)$$

So we want $\frac{r^2}{1-r} = \frac{1}{20}$. And

$$\frac{r^2}{1 - r} = \frac{1}{20} \quad \Leftrightarrow \quad 20r^2 = 1 - r \quad \Leftrightarrow \quad 20r^2 + r - 1 = 0 \ .$$

So we want

$$r \ = \ \frac{-1 \pm \sqrt{1 - (4)(20)(-1)}}{2\,(20)} \ = \ \frac{-1 \pm \sqrt{81}}{40} \ = \ \frac{-1 \pm 9}{40} \ = \ \begin{cases} \frac{-1+9}{40} = \frac{8}{40} = \frac{1}{5} & \text{for the } + \\ \frac{-1-9}{40} = \frac{-10}{40} = \frac{-1}{4} & \text{for the } - \end{cases} .$$

Note that $\left|\frac{1}{5}\right| < 1$ and $\left|\frac{-1}{4}\right| < 1$, as was needed for (1) So there are 2 (distinct) values of r, namely $r = \frac{1}{5}$ and $r = \frac{-1}{4}$, such that $\sum_{n=2}^{\infty} r^n \ = \ \frac{1}{20}$.

13. Consider the following two series.

$$\text{Series A is} \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\text{Series B is} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} .$$

13soln. The series $\sum \frac{1}{n}$ is a p-series of the form $\sum \frac{1}{n^p}$ with $p = 1 \leq 1$ so $\sum \frac{1}{n}$ diverges. Now consider $\sum \frac{(-1)^n}{n} = \sum (-1)^n u_n$ where $u_n = \frac{1}{n}$. Clearly,

(1) $\frac{1}{n} > 0$
(2) $\lim_{n\to\infty} \frac{1}{n} = 0$
(3) $\frac{1}{n} \geq \frac{1}{n+1}$.

So by the Alternating Series Test (AST), $\sum \frac{(-1)^n}{n}$ converges. Thus $\sum \frac{(-1)^n}{n}$ is conditionally convergent since $\sum \frac{(-1)^n}{n}$ converges but $\sum \left| \frac{(-1)^n}{n} \right|$ diverges.

14. The formal series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} .$$

14soln. converges conditionally as can be shown by using the LCT with $b_n = \frac{1}{n}$ as well as the AST.

Abs. Conv? Consider $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right| = \sum \frac{1}{\sqrt{(n+2)(n+7)}}$

Thinking Land: $a_n = \frac{1}{\sqrt{(n+2)(n+7)}} \overset{n \text{ big}}{\approx} \frac{1}{\sqrt{n \cdot n}} = \frac{1}{n} = b_n$

LCT. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{(n+2)(n+7)}} = 1$

more details → $= \lim_{n\to\infty} \sqrt{\left[\frac{n^2}{(n+2)(n+7)} \right]} = \sqrt{1} = 1 = L$, $0 < L < \infty$

so $\sum b_n$ & $\sum a_n$ do the same thing. $\sum b_n$ divg (harmonic series)

so $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right|$ dvg so not abs. conv.

Cond. Conv? Let's use AST w/ $0 \leq u_n = \frac{1}{\sqrt{(n+2)(n+7)}}$

(1) u_n dec, ie $u_n > u_{n+1}$? yes clear.

(2) $\lim_{n\to\infty} \frac{1}{\sqrt{(n+2)(n+7)}} = 0$ ☺

so, by AST, $\sum (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}}$ conv.

15. Consider the formal series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = (-1)^n \frac{(n+1)!}{(2n)!}$$

and let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

15soln. Below we show $\rho = 0$. Thus $\sum (-1)^n \frac{(n+1)!}{(2n)!}$ is absolutely convergent (by the Ratio Test).

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+2)!}{(2n+2)!}\,\frac{(2n)!}{(n+1)!} = \frac{(n+2)!}{(n+1)!}\,\frac{(2n)!}{(2n+2)!} = \frac{n+2}{(2n+1)(2n+2)} = \frac{\frac{n+2}{n^2}}{\frac{(2n+1)}{n}\,\frac{(2n+2)}{n}}$$

$$= \frac{\frac{1}{n}+\frac{2}{n^2}}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)} \xrightarrow{n\to\infty} \frac{0+0}{(2+0)\,(2+0)} = 0 \stackrel{\text{def}}{=} \rho .$$

Thus $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ and so $\sum (-1)^n \frac{(n+1)!}{(2n)!}$ is absolutely convergent by the Ratio Test.

16. What is the LARGEST interval for which the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

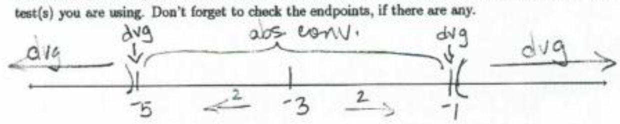
is absolutely convergent?

16soln. The LARGEST interval for which the series $\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$ is absolutely convergent is $(-5,-1)$.

Hint: $(2x+6)^n = [2(x+3)]^n = 2^n (x+3)^n = 2^n (x-(-3))^n$

The center is $x_0 =$ -3 and the radius of convergence is $R =$ 2.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



Ratio Test $\rho = \lim_{n\to\infty} \left| \frac{(2x+6)^{n+1}}{4^{n+1}} \frac{4^n}{(2x+6)^n} \right| = \lim_{n\to\infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$\updownarrow$ or

Root Test. $\rho = \lim_{n\to\infty} \left| \frac{(2x+6)^n}{4^n} \right|^{1/n} = \lim_{n\to\infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$\rho < 1 \iff |2x+6| < 4 \iff 2|x+3| < 4 \iff |x+3| < 2 \iff |x-(-3)| < 2$

endpts. $-3+2 = -1$ and $-3-2 = -5$

Check endpts

$x = -1$: $\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1 = 1+1+1+1+\cdots = \infty$ dvg

$x = -5$ $\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1\cdot 4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n}$

$= \sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+1-1+\cdots$

osc btw -1 & 0 $\Rightarrow$ dvg

17. Suppose that the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is 16. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{2n}$?

17soln. We are given that

$$\begin{aligned} &\sum c_n x^n \quad \text{is absolutely convergent when} \quad |x| < 16 \\ &\sum c_n x^n \quad \text{is divergent when} \quad |x| > 16\ . \end{aligned}$$

Thus

$$\begin{aligned} &\sum c_n x^{2n} \overset{\text{note}}{=} \sum c_n \left(x^2\right)^n \quad \text{is absolutely convergent when} \quad \left|x^2\right| < 16 \text{ ,i.e., when } |x| < 4 \\ &\sum c_n x^{2n} \overset{\text{note}}{=} \sum c_n \left(x^2\right)^n \quad \text{is divergent when} \quad \left|x^2\right| > 16 \text{ ,i.e., when } |x| > 4\ . \end{aligned}$$

Thus the radius of convergence of the power series $\sum c_n x^{2n}$ is 4.

18. In class we learned that, for each $x \in \mathbb{R}$,

$$\cos x \;=\; \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!}\ .$$

Use this to find a Taylor expansion about the center $x_0 = 0$ (i.e., Maclaurin series) for

$$f(x) = x\cos(4x)\ .$$

18soln. $x\cos(4x) \;=\; x \sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n}}{(2n)!} \;=\; x^1 \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n} x^{2n}}{(2n)!} \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n} x^{2n+1}}{(2n)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{16^n x^{2n+1}}{(2n)!}$

19. Find the Taylor series for $f(x) = (1-5x)^{-3}$ about the center $x_0 = 0$.

19soln. The computations below show that the Maclaurin series for $f(x) = (1-5x)^{-3}$ is $\sum_{n=0}^{\infty} \frac{5^n (n+1)(n+2)}{2} x^n$.

we were given $x_0 = 0$				pattern search for $\frac{f^{(n)}(x_0)}{n!}$
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$\frac{f^{(n)}(x_0)}{n!}$	
0	$(1-5x)^{-3}$	1	1	$5^0\ \frac{(0+1)(0+2)}{2}$
1	$(-3)(1-5x)^{-4}(-5) =$ $(5)(3)(1-5x)^{-4}$	$5(3)$	$5\ \frac{(3)}{1!}$	$5^1\ \frac{(1+1)(1+2)}{2}$
2	$(5)(3)(-4)(1-5x)^{-5}(-5) =$ $5^2(3)(4)(1-5x)^{-5}$	$5^2(3)(4)$	$5^2\ \frac{(3)(4)}{2!}$	$5^2\ \frac{(2+1)(2+2)}{2}$
3	$5^2(3)(4)(-5)(1-5x)^{-6}(-5) =$ $5^3(3)(4)(5)(1-5x)^{-6}$	$5^3(3)(4)(5)$	$5^3\ \frac{(3)(4)(5)}{3!}$	$5^3\ \frac{(3+1)(3+2)}{2}$
4	$5^3(3)(4)(5)(-6)(1-5x)^{-7}(-5) =$ $5^4(3)(4)(5)(6)(1-5x)^{-7}$	$5^4(3)(4)(5)(6)$	$5^4\ \frac{(3)(4)(5)(6)}{4!}$	$5^4\ \frac{(4+1)(4+2)}{2}$
n	$5^n\,[(3)(4)(5)\cdots(n+2)]\,(1-5x)^{-(n+3)}$	$5^n\,[(3)(4)(5)\cdots(n+2)]$	$5^n\ \frac{(3)(4)(5)\cdots(n+2)}{n!}$	$5^n\ \frac{(n+1)(n+2)}{2}$

Pattern search for $\frac{f^{(n)}(x_0)}{n!}$:

$$\begin{aligned} \frac{f^{(n)}(x_0)}{n!} &= 5^n\, \frac{(3)(4)(5)\cdots(n+2)}{n!} \;=\; 5^n\, \frac{1}{2}\, \frac{(2)(3)(4)(5)\cdots(n+2)}{n!} \;=\; 5^n\, \frac{1}{2}\, \frac{(n+2)!}{n!} \\ &= 5^n\, \frac{1}{2}\, \frac{(n)!\,(n+1)(n+2)}{n!} \;=\; 5^n\, \frac{(n+1)(n+2)}{2} \end{aligned}$$

20. Consider the function $f(x) = e^x$ over the interval $(-1,3)$. The 4^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_4(x) \;=\; 1 \;+\; x \;+\; \frac{x^2}{2!} \;+\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;=\; \sum_{n=0}^{4} \frac{x^n}{n!}\ .$$

The 4^{th} order Remainder term $R_4(x)$ is defined by $R_4(x) = f(x) - P_4(x)$ and so $e^x \approx P_4(x)$ where the approximation is within an error of $|R_4(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_4(x)|$ that is valid for each $x \in (-1, 3)$.

20soln. By Taylor's Remainder Theorem, for each $x \in (-1, 3)$, there exists c between x and 0 so that

$$R_4(x) = \frac{f^{(5)}(c)\ (x-0)^5}{5!}\ .$$

Note that if $x \in (-1, 3)$ and c is between x and 0, then $c \in (-1, 3)$. Also, $\left|f^{(5)}(x)\right| = e^x$.
So for each $x \in (-1, 3)$,

$$|R_4(x)| = \left|\frac{f^{(5)}(c)\ (x-0)^5}{5!}\right| = \frac{\left|f^{(5)}(c)\right|\ |x|^5}{5!} = \frac{e^c\ |x|^5}{5!} \le \frac{e^c\ 3^5}{5!} \le \frac{e^3\ 3^5}{5!}\ .$$

21. The point with a polar coordinate representation $\left(-1, \frac{\pi}{4}\right)$ also has a polar coordinate representation $(\ {}^{+}1, \theta\ )$ where θ is

21soln.

A point with polar coordinates (r, θ) has Cartesian coordinates $(r\cos\theta, r\sin\theta)$.
So we want to find θ so that

$$(+1)\cos\theta \overset{\text{want}}{=} (-1)\cos\frac{\pi}{4} \overset{\text{know}}{=} \frac{-\sqrt{2}}{2}$$

and

$$(+1)\sin\theta \overset{\text{want}}{=} (-1)\sin\frac{\pi}{4} \overset{\text{know}}{=} \frac{-\sqrt{2}}{2}$$

So we can take $\theta = \frac{5\pi}{4}$.

22. Find $\dfrac{dy}{dx}$ for the parameterized curve given by

$$\begin{aligned} x &= 2t^2 + 1 \\ y &= 3t^3 + 2 \end{aligned}$$

for ${}^{-}17 \le t \le 17$.

22soln.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t^2}{4t} = \frac{9t}{4}$$

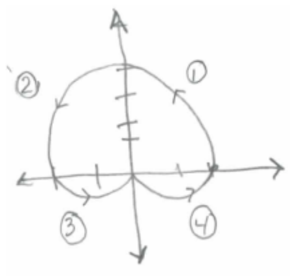
23. Express the area enclosed by the curve given by (in polar coordinates)

$$r = 2 + 2\sin\theta$$

as an integral.

helpful table for graphing				
period of $\sin(\mathbf{1}\theta) = \frac{2\pi}{\mathbf{1}} = 2\pi$ and so $\frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2} = \Delta\theta$				
	θ	$\sin\theta$	$2\sin\theta$	$r = 2 + 2\sin\theta$
①	$0 \to \frac{\pi}{2}$	$0 \to 1$	$0 \to 2$	$2 \to 4$
②	$\frac{\pi}{2} \to \pi$	$1 \to 0$	$2 \to 0$	$4 \to 2$
③	$\pi \to \frac{3\pi}{2}$	$0 \to -1$	$0 \to -2$	$2 \to 0$
④	$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$-2 \to 0$	$0 \to 2$

23soln.



The curve encloses an area (looks like an upside down heart) once as θ goes from 0 to 2π.
So let $\alpha = 0$ and $\beta = 2\pi$
The area of a sector (of a circle with radius r determined by an angle θ) is $\frac{1}{2}r^2\theta$.
So the desired area, expressed as an integral, is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \boxed{\frac{1}{2}\int_{0}^{2\pi} (2 + 2\sin\theta)^2\, d\theta}\,.$$

24. Express the arc length of the *heart* traced out by the curve given by (in polar coordinates)

$$r = 2 + 2\sin\theta$$

as an integral.

24soln. See previous problem for graph. Note the heart is traced out once as θ goes from 0 to 2π.
So let $\alpha = 0$ and $\beta = 2\pi$
So the arc length, expressed as an integral, is

$$AL = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \boxed{\int_{0}^{2\pi} \sqrt{(2 + 2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta}$$

25. Prof. Girardi likes

ⓐ moose
ⓑ colored chalk
ⓒ peanut butter
ⓓ mathematics
ⓔ All of the above.

Good Luck in your math fun to come!