MARK BOX		
PROBLEM	POINTS	
0	20	
1-7	70	
8	10	
%	100	

NAME: Work Moose

PIN: 17

INSTRUCTIONS

- **On Problem 0**, fill in the blanks/boxes. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For multiple choice problems**, circle your answer(s) on the provided chart. No need to show work.
- **For all other problems**, to receive credit you **MUST** show ALL your work and :
  1. **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  2. if a line/box is provided, then:
     — show you work BELOW the line/box
     — put your answer on/in the line/box
  3. if no such line/box is provided, then box your answer.
- Upon request, you will be given as much (blank) scratch paper as you need.
- Check that your copy of the exam has all of the problems.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: electronic devices, books, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a students request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.7–10.10 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in-the blanks/boxes.

0A. Power Series Consider the (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \,(x - x_0)^n \;, \tag{0-1}$$

with radius of convergence $R \in [0, \infty]$.
(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)
Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

- First let $0 < R < \infty$. The largest set of x's for which we know that the power series in (0-1) is:
  (a) absolutely convergent is $(x_0 - R, x_0 + R)$, also ok: $\{x \in \mathbb{R} \colon \; |x - x_0| < R\}$
  (b) divergent is $(-\infty, x_0 - R) \cup (x_0 + R, \infty)$, also ok: $\{x \in \mathbb{R} \colon \; |x - x_0| > R\}$.

  What can you say about the convergence of the power series in (0-1) when $x = x_0 + R$ or $x = x_0 - R$?

  the series can be doing anything, i.e., there are examples showing that it can be absolutely convergent, conditionally convergent or divergent

- Now let $R = \infty$. The largest set of x's for which we know that the power series in (0-1) is:
  (a) absolutely convergent is $\mathbb{R}$, also ok: $\{x \in \mathbb{R} \colon \; |x - x_0| < R\}$
  (b) divergent is $\emptyset$, also ok: the empty set .

- Now let $R = 0$. The largest set of x's for which we know that the power series in (0-1) is:
  (a) absolutely convergent is $\{x_0\}$, also ok: $\{x \in \mathbb{R} \colon x = x_0\}$
  (b) divergent is $(-\infty, x_0) \cup (x_0, \infty)$, also ok: $\{x \in \mathbb{R} \colon x \neq x_0\}$ or $\mathbb{R} \setminus \{x_0\}$.

- Now let $R > 0$ and fill-in the 5 boxes.
  Consider the function $y = h(x)$ defined by the power series in (0-1).
  (a) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

  $$h'(x) \;=\; \sum_{n=1}^{\infty} \boxed{n \, a_n \,(x - x_0)^{n-1}} \;. \tag{0-2}$$

  What can you say about the radius of convergence of the power series in (0-2)? $\boxed{\text{It's the same } R}$.
  (b) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

  $$\int_{x=\alpha}^{x=\beta} h(x)\,dx \;=\; \sum_{n=0}^{\infty} \boxed{\frac{a_n}{n+1}\,(x - x_0)^{n+1}} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} \;.$$

0B. Taylor/Maclaurin Polynomials and Series. Fill-in the boxes.

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. In open form (i.e., with "…" notation and without a $\sum$-sign)

$$P_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N$$

b. In closed form (i.e., with a $\sum$-sign and without "…" notation)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

c. In open form (i.e., with "…" notation and without a $\sum$-sign)

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$$

d. In closed form (i.e., with a $\sum$-sign and without "…" notation)

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

e. The formula for c_n is

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{(N+1)}$$ for some c between x and x_0.

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = 0$.

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS

- See **Statement of Multiple Choice Problems** for the statement of the multiple choice.
- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 10 points.
  - For a problem with precisely two answers marked, one of which is correct, 4 points.
  - For a problem with nothing marked (i.e., left blank) 1 point.
  - All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Your Solutions							Do Not Write Below			
							points			
PROBLEM						number of solutions circled	10	4	1	0
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
							Extra Credit:			

8. In this problem, you must show your work. Clearly explain your thought process. Using Taylor's (BIG) Remainder Theorem, show that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \;=\; e\,.$$

Hint: The instruction says to use Taylor's Remainder Theorem so you cannot use the facts listed on the *Commonly Used Taylor Series* handout. (Indeed, the facts listed on the *Commonly Used Taylor Series* handout are shown by using Taylor's Remainder Theorem so think of this problem as showing one of these facts.)
Hint. Consider the function $f(x) = e^x$.

Let $f(x) = e^x$ with $f\colon \mathbb{R} \to \mathbb{R}$. Take the center $x_0 = 0$. Let's follow the notation from problem **0B**. So $f^{(n)}(x) = e^x$ and $f^{(n)}(x_0) = e^x$ for each $n \in \mathbb{N} \cup \{0\}$. So

$$e^x = P_N(x) + R_N(x)$$

and

$$e^x = P_\infty(x) \qquad \text{if and only if} \qquad \lim_{N\to\infty} |R_N(x)| = 0$$

where

$$P_N(x) \;=\; \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \;=\; \sum_{n=0}^{N} \frac{x^n}{n!}$$

and

$$P_\infty(x) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and Taylor's Remainder Theorem tells us that

$$\text{for some } c \text{ between } x \text{ and } 0: \qquad R_N(x) \;=\; \frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)} \;=\; \frac{e^c x^{N+1}}{(N+1)!} \tag{R}$$

Taking $x = 1$ in (R) gives

$$\text{for some } c \text{ between } 1 \text{ and } 0: \qquad |R_N(1)| \;=\; \left|\frac{e^c 1^{N+1}}{(N+1)!}\right| \overset{0\le c\le 1}{\le} \frac{e^1}{(N+1)!} \xrightarrow{N\to\infty} 0\,.$$

So $e^1 = P_\infty(1)$, i.e.,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}\,.$$

Statement of Multiple Choice 1–7
and
Scratch Paper
Not to be collected.

1. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) = \frac{2}{3-x}$$

about the center $x_0 = 0$ and state when this Taylor series is valid. Hint, by simple algebra,

$$f(x) = \frac{2}{3-x} = \left(\frac{2}{3}\right)\left(\frac{1}{1-\frac{x}{3}}\right).$$

a. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n x^n$, valid for $|x| < 1$
b. $\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n$, valid for $|x| < 3$
c. $\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n$, valid for $|x| < 1$
d. $\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n$, valid for $|x| < 3$
e. none of these

We know the Geometric Series (a Commonly Used Taylor Series): $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$, valid for $|r| < 1$.
So

$$f(x) = \frac{2}{3-x} = \left(\frac{2}{3}\right)\left[\frac{1}{1-\frac{x}{3}}\right] \overset{\text{GS}}{=} \left(\frac{2}{3}\right)\left[\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n\right] = \sum_{n=0}^{\infty}\left(\frac{2}{3^1}\right)\left(\frac{x^n}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{2}{3^{n+1}}\right)x^n.$$

And GS is valid $\iff \left|\frac{x}{3}\right| < 1 \iff |x| < 3$

2. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) = \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$. Hint. Start with the Taylor series expansion

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{valid for } |x| < 1$$

and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} = (-1)(1-x)^{-2}\,D_x(1-x) = (-1)(1-x)^{-2}\,(-1) = (1-x)^{-2}.$$

a. $\sum_{n=0}^{\infty} \frac{(n)(n-1)(n-2)}{6} x^{n-3}$
b. $\sum_{n=0}^{\infty} (n)(n-1)(n-2)\, x^n$
c. $\sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6} x^n$
d. $\sum_{n=0}^{\infty} (-1)^n \frac{(n+3)(n+2)(n+1)}{6} x^n$
e. none of these

2. Start with Geometric Series and take Derivatives as many times as need. Geometric Series is valid when $|x| < 1$ so resulting power series expansions will also be valid when $|x|<1$.

Geometric Series $\Rightarrow (1-x)^{-1} = \sum_{k=0}^{\infty} x^k \overset{D_x}{\Longrightarrow} (1-x)^{-2} = \sum_{k=1}^{\infty} k x^{k-1}$

$\overset{D_x}{\Longrightarrow} 2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1) x^{k-2} \overset{D_x}{\Longrightarrow} 2\cdot 3(1-x)^{-4} = \sum_{k=3}^{\infty} k(k-1)(k-2) x^{k-3}$

So

$$(1-x)^{-4} = \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{6} x^{k-3} \overset{4}{=} \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6} x^n$$

let $k-3=n$ $\Rightarrow k = n+3$

3. Using a known (commonly used) Taylor series, evaluate $\int \tan^{-1}(t^2)\,dt$ as a power series.

a. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+3}}{(4n+3)}$

b. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+3}}{(2n+1)(4n+3)}$

c. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+2}}{(2n+3)}$

d. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+2}}{(2n+1)}$

e. $C + \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+3}}{(2n+3)}$

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$$\int \tan^{-1}(t^2)\,dt = \int \left[\sum_{k=1}^{\infty} (-1)^{k-1} \frac{(t^2)^{2k-1}}{2k-1} \right] dt$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \left[\int t^{4k-2}\,dt \right] = C + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)} \frac{t^{4k-1}}{4k-1}$$

(note $k=1 \Leftrightarrow k-1=0$ so let $k-1=n$ and so $k=n+1$)

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} t^{4n+3}.$$

Problem Source: Practice Exam 3 # 2.

4. Find the 3rd order Taylor polynomial for $f(x) = \frac{1}{x}$ about the center $x_0 = 2$.

a. $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

b. $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}(x-2)^2 - \frac{3}{8}(x-2)^3$

c. $\frac{1}{2} + \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 + \frac{1}{16}(x-2)^3$

d. $\frac{1}{2} - \frac{1}{4}x + \frac{1}{4}x^2 - \frac{3}{8}x^3$

e. none of these

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-1}	$1/2$
1	$-x^{-2}$	$-1/4$
2	$2x^{-3}$	$+1/4$
3	$-6x^{-4}$	$-3/8$

$$P_3(x) = \frac{f^{(0)}(2)}{0!} + \frac{f^{(1)}(2)}{1!}(x-2)^1 + \frac{f^{(2)}(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}\cdot\frac{1}{2}(x-2)^2 - \frac{3}{8}\cdot\frac{1}{2\cdot 3}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

5. Find the Taylor series for $f(x) = x^4 - 3x^2 + 1$ about the center $x_0 = 1$.

a. $(x-1)^4 - 3(x-1)^2 + 1$
b. $-1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$
c. $-1 - 2(x-1) + 6(x-1)^2 + 24(x-1)^3 + 24(x-1)^4$
d. $-1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4 + 2(x-1)^5$
e. none of these

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}$
0	$x^4 - 3x^2 + 1$	-1	-1
1	$4x^3 - 3\cdot 2x$	-2	-2
2	$4\cdot 3x^2 - 3\cdot 2$	$6 = 3\cdot 2$	3
3	$4\cdot 3\cdot 2x$	$4\cdot 3\cdot 2$	4
4	$4\cdot 3\cdot 2$	$4\cdot 3\cdot 2$	1
5	0	0	0
$n \geq 5$	0	0	0

So $p_\infty(x) = -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + 1(x-1)^4 + 0(x-1)^5 + 0(x-1)^6 + 0(x-1)^7 + \ldots$.

6. Find the Taylor series for

$$f(x) = \frac{1}{x^2}$$

about the center $x_0 = 1$.

a. $\sum_{n=0}^{\infty} (-1)^n (n+1)!\, x^n$
b. $\sum_{n=0}^{\infty} (-1)^n (n+1)!\, (x-1)^n$
c. $\sum_{n=0}^{\infty} (-1)^n (n+1)\, (x-1)^n$
d. $\sum_{n=0}^{\infty} (-1)^{n+1} (n+1)\, (x-1)^n$
e. none of these

n	$f^{(n)}(x)$	$f^{n}(1)$	$\frac{f^{n}(1)}{n!}$	note
0	x^{-2}	1	1	$1 = (-1)^0 (0+1)$
1	$-2x^{-3}$	-2	-2	$-2 = (-1)^1 (1+1)$
2	$+2\cdot 3\, x^{-4}$	$+3!$	3	$3 = (-1)^2 (2+1)$
3	$-2\cdot 3\cdot 4\, x^{-5}$	$-4!$	-4	$-4 = (-1)^3 (3+1)$
4	$+2\cdot 3\cdot 4\cdot 5\, x^{-6}$	$5!$	$+5$	$5 = (-1)^4 (4+1)$
$n > 4$			$(-1)^n (n+1)$	↑ checking

$$p_\infty(x) = \sum_{n=0}^{\infty} (-1)^n (n+1) (x-1)^n$$

7. Consider the function

$$f(x) = e^{-x} .$$

The 5^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_5(x) = \sum_{n=0}^{5} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} .$$

The 5^{th} order Remainder term $R_5(x)$ is defined by $R_5(x) = f(x) - P_5(x)$ and so $e^{-x} \approx P_5(x)$ where the approximation is within an error of $|R_5(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_5(x)|$ that is valid for each $x \in (7, 9)$.

a. $\frac{(e^{9})(9^5)}{5!}$ b. $\frac{(e^{-9})(9^6)}{6!}$ c. $\frac{(e^{-7})(9^6)}{6!}$ d. $\frac{(e^{-0})(9^6)}{6!}$ e. none of these

By Taylor's Remainder Theorem, for each $x \in (7, 9)$, there exists c between x and 0 so that

$$R_5(x) = \frac{f^{(6)}(c)\ (x-0)^6}{6!} .$$

Note that if $x \in (7, 9)$ and c is between x and 0, then $c \in (0, 9)$. So for each $x \in (7, 9)$,

$$|R_5(x)| = \left| \frac{f^{(6)}(c)\ (x-0)^6}{6!} \right| = \frac{\left|f^{(6)}(c)\right|\ |x|^6}{6!} = \frac{e^{-c}\ |x|^6}{6!} \le \frac{e^{-c}\ 9^6}{6!} \le \frac{e^{-0}\ 9^6}{6!} = \frac{9^6}{6!} .$$

If you prefer, you can also think of the above line as:

$$|R_5(x)| = \left| \frac{f^{(6)}(c)\ (x-0)^6}{6!} \right| = \frac{\left|f^{(6)}(c)\right|\ |x|^6}{6!} = \frac{e^{-c}\ |x|^6}{6!} = \frac{|x|^6}{e^c\ 6!} \le \frac{9^6}{e^c\ 6!} \le \frac{9^6}{e^0\ 6!} = \frac{9^6}{6!} .$$