MARK BOX		
PROBLEM	POINTS	
0	20	
1	20	
2	10	
3	10	
4-7	20	
8	10	
9	10	
%	100	

NAME: KEY-e-poo

PIN: 17

INSTRUCTIONS

- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For multiple choice problems**, circle your answer(s) on the provided chart. No need to show work.
- **For all other problems**, to receive credit you **MUST** show ALL your work and :
  (1) **work in a logical fashion, show all your work, indicate your reasoning**;
  no credit will be given for an answer that *just appears*;
  such explanations help with partial credit
  (2) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (3) if no such line/box is provided, then box your answer.
- Upon request, you will be given as much (blank) scratch paper as you need.
- Check that your copy of the exam has all of the problems.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: electronic devices, books, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a students request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.1–10.6 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in-the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0.1. Geometric Series. Fill in the boxes with the proper range of $r \in \mathbb{R}$.

- The series $\sum r^n$ converges if and only if r satisfies $\boxed{|r| < 1}$.

0.2. p-series. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if $\boxed{p > 1}$.

0.3. State the **Integral Test** for a positive-termed series $\sum a_n$.
Let $f\colon [1, \infty) \to \mathbb{R}$ be so that

- $a_n = f(n)$ for each $n \in \mathbb{N}$
- f is a $\boxed{\text{positive}}$ function
- f is a $\boxed{\text{continuous}}$ function
- f is a $\boxed{\text{decreasing (nonincreasing is also ok)}}$ function.

Then $\sum a_n$ converges if and only if $\boxed{\int_{x=1}^{x=\infty} f(x)\, dx}$ converges.

0.4. State the **Comparison Test** for a positive-termed series $\sum a_n$. Let $N_0 \in \mathbb{N}$ (e.g., N_0 might be 17).

- If $\boxed{0 \le a_n \le c_n}$ (only $a_n \le c_n$ is also ok b/c given $a_n \ge 0$) when $n \ge N_0$ and $\boxed{\sum c_n \text{ converges}}$, then $\sum a_n$ converges.
- If $\boxed{0 \le d_n \le a_n}$ (need $0 \le d_n$ part here) when $n \ge N_0$ and $\boxed{\sum d_n \text{ diverges}}$, then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.5. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.
Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then $\boxed{[\, \sum b_n \text{ converges} \iff \sum a_n \text{ converges} \,]}$.

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.6. By definition, for an arbitrary series $\sum a_n$, (fill in these 3 boxes with convergent or divergent).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ is $\boxed{\text{convergent}}$.
- $\sum a_n$ is conditionally convergent if and only if
  $\sum a_n$ is $\boxed{\text{convergent}}$ and $\sum |a_n|$ is $\boxed{\text{divergent}}$.
- $\sum a_n$ is divergent if and only if $\sum a_n$ is divergent.

0.7. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$.
Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \ .$$

- If $\boxed{\rho < 1}$ then $\sum a_n$ converges absolutely.
- If $\boxed{\rho > 1}$ then $\sum a_n$ diverges.
- If $\boxed{\rho = 1}$ then the test is inconclusive.

0.8. State the **Alternating Series Test (AST)**.
If (1) $u_n > 0$ for each $n \in \mathbb{N}$ (2) $\lim_{n\to\infty} u_n = \boxed{0}$ (3) u_n $\boxed{> \text{ (also ok } \ge)}$ u_{n+1} for each $n \in \mathbb{N}$,
then $\boxed{\text{the series } \sum (-1)^n u_n \text{ converges. (also ok: } \sum(-1)^{n+1} u_n \text{ converges or } \sum(-1)^{n-1} u_n \text{ converges)}}$

1. Circle T if the statement is TRUE. Circle F if the statement if FALSE. To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true. Scoring: 2 pts for correct answer, 0 pts for an incorrect answer, 1 pt for a blank answer (indicated by a circled B).

On the next 3, think of the n^{th}-term test for divergence and what if $a_n = \frac{1}{n}$			
T	F	B	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
T	F	B	If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
T	F	B	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
On the next 5, think of AC vs. CC vs. Divergent. Examples from Problem 2 might be helpful.			
T	F	B	A series $\sum a_n$ is precisely one of the following: absolutely convergent, conditionally convergent, divergent.
T	F	B	If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ is either absolutely convergent or divergent.
T	F	B	If $\sum |a_n|$ converges, then $\sum a_n$ converges.
T	F	B	If $\sum\ a_n$ diverges, then $\sum |a_n|$ diverges.
T	F	B	If $\sum |a_n|$ diverges, then $\sum a_n$ diverges.
On the next 2, think of a Theorem from class and what if $b_n = -a_n$.			
T	F	B	If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.
T	F	B	If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.

2. Circle the behavior of the given series. The abbreviations are:

- AC stands for absolutely convergent
- CC stands for conditionally convergent
- DVG stand for divergent
- NOT stands for none of the others.

You can circle up to **1** answers for each problem. The scoring is as follows.

- For a problem with precisely one answer marked and the answer is correct, 1 points.
- All other cases, 0 points.

Series				
$\sum_{n=1}^{\infty} \frac{1}{n^2}$	AC	CC	DVG	NOT
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	AC	CC	DVG	NOT
$\sum_{n=1}^{\infty} \frac{1}{n}$	AC	CC	DVG	NOT
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$	AC	CC	DVG	NOT
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	AC	CC	DVG	NOT
$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	AC	CC	DVG	NOT
$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$	AC	CC	DVG	NOT
$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$	AC	CC	DVG	NOT
$\sum_{n=1}^{\infty} \frac{1}{e^n}$	AC	CC	DVG	NOT
$\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$	AC	CC	DVG	NOT

3. For the following **SEQUENCES**:

- if the limit exists, find it
- if the limit does not exist, then say that it DNE (which is equivalent to saying it diverges).

Put your ANSWER IN the box and show your WORK BELOW the box.

3a. $\displaystyle \lim_{n\to\infty} \frac{5n^2+4\sqrt{n}}{6n^2+7n+1} \;=\; \frac{5}{6}$

$$\lim_{n\to\infty} \frac{\frac{5n^2}{n^2}+\frac{4n^{1/2}}{n^2}}{\frac{6n^2}{n^2}+\frac{7n}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5+\frac{4}{n^{3/2}}}{6+\frac{7}{n}+\frac{1}{n^2}} = \frac{5}{6}$$

3b. $\displaystyle \lim_{n\to\infty} \frac{-5n^8+4\sqrt{n}}{6n^3+7n^2+1} \;=\; -\infty$ (Also correct, but not as informative, is DNE.)

$$\lim_{n\to\infty} \frac{\frac{-5n^8}{n^8}+\frac{4n^{1/2}}{n^8}}{\frac{6n^3}{n^8}+\frac{7n^2}{n^8}+\frac{1}{n^8}} = \lim_{n\to\infty} \frac{-5+\frac{4}{n^{15/2}}}{\underbrace{\cancel{\tfrac{6}{n^5}}}_{0}+\underbrace{\cancel{\tfrac{7}{n^6}}}_{0}+\underbrace{\cancel{\tfrac{1}{n^8}}}_{0}} = \cancel{\frac{-5}{0}} = -\infty \text{ or DNE}$$

3c. $\displaystyle \lim_{n\to\infty} \frac{5n^3+4\sqrt{n}}{6n^8+7n^2+1} \;=\; 0$

$$\lim_{n\to\infty} \frac{\frac{5n^3}{n^8}+\frac{4n^{1/2}}{n^8}}{\frac{6n^8}{n^8}+\frac{7n^2}{n^8}+\frac{1}{n^8}} = \lim_{n\to\infty} \frac{\overbrace{\frac{5}{n^5}}^{\to 0}+\overbrace{\frac{4}{n^{15/2}}}^{\to 0}}{6+\underbrace{\frac{7}{n^6}}_{\to 0}+\underbrace{\frac{1}{n^8}}_{\to 0}}$$

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS

- See **Statement of Multiple Choice Problems** for the statement of the multiple choice.
- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 5 points.
  - For a problem with precisely two answers marked, one of which is correct, 3 points.
  - For a problem with nothing marked (i.e., left blank) 1 point.
  - All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Your Solutions							Do Not Write Below			
							points			
PROBLEM						number of solutions circled	5	3	1	0
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
							Extra Credit:			

8. In this problem, you must show your work. Let

$$a_n = \frac{3^n}{n!}$$

8a. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{3}{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{n!(3^{n+1})}{(n+1)!(3^n)} = \frac{\cancel{n!}\cancel{(3^n)}(3^1)}{\cancel{n!}(n+1)\cancel{(3^n)}} = \frac{3}{n+1}$$

8b. Check the correct box and then indicate your reasoning below. **SHOW ALL YOUR WORK**. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

- [x] absolutely convergent
- [ ] ~~conditionally convergent~~ positive termed series
- [ ] divergent

ratio Test

by 8a.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \xrightarrow{n \to \infty} 0$$

$\rho = 0 < 1$ Converges

So by the Ratio Test $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ absolutely Converges

9. Check the correct box and then indicate your reasoning below. **SHOW ALL YOUR WORK.** Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

Spring 2016 Beautiful ! Exam 2

9. Check the correct box and then indicate your reasoning below. **SHOW ALL YOUR WORK.** Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n^5}}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

For $\boxed{q>0}$, n big enough:

$\ln(n) \leq n^{q} \leq e^{n}$

$\frac{\ln(n)}{n^{5/2}} \leq \frac{n^{q}}{n^{5/2}}$

Guess: bound above by cvg = cvg

Need $p>1$

$\frac{n^{q}}{n^{5/2}} = \frac{1}{n^{5/2-q}}$ $\qquad \frac{5}{2}-q>1$

$-q > -\frac{3}{2}$

$\boxed{q < \frac{3}{2}}$

choose $q=1$

$\sum \frac{\ln(n)}{\sqrt{n^5}} \leq \sum \frac{1}{n^{5/2-1}}$

$\sum \frac{\ln(n)}{\sqrt{n^5}} \leq \sum \frac{1}{n^{3/2}}$

by p-series, $p=\frac{3}{2}>1$, so $\sum \frac{1}{n^{3/2}}$ converges,

and since $\sum \frac{\ln(n)}{\sqrt{n^5}}$ is bound above by convergent, $\sum \frac{\ln(n)}{\sqrt{n^5}}$ is convergent.

Also, $\sum_{n=1}^{\infty} \left|\frac{\ln n}{\sqrt{n^5}}\right| = \sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n^5}}$, so CC is not an option

Statement of Multiple Choice 4–7
and
Scratch Paper
Not to be collected.

4. Consider the following two series.

Series A is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Series B is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

a. both series converge absolutely

b. both series diverge

c. series A converges conditionally and series B diverges

d. series A diverges and series B converges conditionally

e. None of the others.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. $\begin{cases} p\text{-series}, & p=1, \quad p \le 1 \\ & \text{or} \\ \text{harmonic series} \end{cases}$

$\displaystyle\sum \frac{(-1)^n}{n}$ conv. by AST, since $\frac{1}{n} > \frac{1}{n+1}$

and $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$

5. Consider the formal series

$$\sum_{n=1}^{\infty} \frac{1}{n\ (n+1)} \qquad (5)$$

and let

$$s_N \ = \ \sum_{n=1}^{N} \frac{1}{n\ (n+1)}\ .$$

Note that the partial fractions decompostion of $\frac{1}{n\ (n+1)}$ is $\frac{1}{n} - \frac{1}{n+1}$.

a. $s_N = 1 - \frac{1}{N+1}$ and the series in (5) converges to 1.

b. $s_N = 1 + \frac{1}{N+1}$ and the series in (5) converges to 1.

c. $s_N = 1 + \frac{1}{N}$ and the series in (5) converges to 1.

d. $s_N = 1 - \frac{1}{N}$ and the series in (5) converges to 1.

e. None of the others.

SECTION 11.2 SERIES |||| 691

EXAMPLE 6 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

(see Section 7.4). Thus we have

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

and so $$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

6. Consider the formal series

$$\sum_{n=2}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n .$$

a. The series converges by the Root Test.

b. The series diverges by the Root Test.

c. The Root Test is inconclusive.

d. The Root Test cannot be applies.

e. None of the others.

EXAMPLE 6 Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

7. The formal series

$$\sum_{n=17}^{\infty} \frac{1}{n \, \ln n}$$

is:

a. convergent by the integral test

b. convergent by the ratio test

c. divergent by the integral test

d. divergent by the ratio test

e. None of the others.

$f(x) = \dfrac{1}{x \ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1 + \ln x}{x^2 (\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} [\ln(\ln x)]_2^t = \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

Basically, apply the integral test. Note that the computation of

$$\int \frac{1}{x \, \ln x} \, dx \;=\; \ln |\ln |u|| \;+\; C \;,$$

as seen by using a u-du subsitution with $u = \ln x$.

Let's see what happens if we try the ratio test. Let $a_n = \frac{1}{n \, \ln n}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \;=\; \lim_{n\to\infty} \frac{1}{(n+1) \, \ln (n+1)} \; \frac{n \, \ln n}{1} \;=\; \lim_{n\to\infty} \frac{n}{n+1} \; \frac{\ln n}{\ln (n+1)} \;.$$

By L'Hopital's Rule,

$$\lim_{n\to\infty} \frac{\ln n}{\ln (n+1)} \;=\; \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} \;=\; \lim_{n\to\infty} \frac{n+1}{n} \;=\; 1 \;.$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the Ratio Test is inconclusive.