MARK BOX		
PROBLEM	POINTS	
0	15	
1-9	45	
10	10	
11	10	
12	10	
13	10	
%	100	

NAME: Solution Key

PIN: 17

INSTRUCTIONS

- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For multiple choice problems 1– 9**, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
- **For problems $\geq$ 10**, to receive credit you **MUST** show ALL your work and :
  (1) **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  (2) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (3) if no such line/box is provided, then box your answer.
- Upon request, you will be given as much (blank) scratch paper as you need.
- Check that your copy of the exam has all of the problems.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: electronic devices, books, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a students request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1–8.5, 8.7, 8.8 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill in the blanks (each worth 1 point).

. If $u \neq 0$, then $\int \frac{du}{u} = \underline{\ln\ |u|} + C$

. $\int \cos u\, du = \underline{\sin u} + C$

. $\int \sin u\, du = \underline{-\cos u} + C$

. $\int \tan u\, du = \underline{\ln|\sec u| \overset{or}{=} -\ln|\cos u|} + C$

. $\int \sec u\, du = \underline{\ln|\sec u + \tan u| \overset{or}{=} -\ln|\sec u - \tan u|} + C$

. $\int \sec^2 u\, du = \underline{\tan u} + C$

. $\int \sec u \tan u\, du = \underline{\sec u} + C$

. Integration by parts formula: $\int u\, dv = \underline{uv - \int v du}$

. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do <u>long division</u>

. Trig. Substitution ($a > 0$): (recall that the *integrand* is the function you are integrating)

▷ if the integrand involves $a^2 + u^2$, then one makes the substitution $u = \underline{a \tan\theta}$

▷ if the integrand involves $a^2 - u^2$, then one makes the substitution $u = \underline{a\ \sin\theta}$

▷ if the integrand involves $u^2 - a^2$, then one makes the substitution $u = \underline{a\ \sec\theta}$

.Trig. Formulas:

▷ $\sin(2\theta) = \underline{2\ \sin\theta\cos\theta}$ (your answer should involve trig functions of θ, and not of 2θ)

▷ $\sin^2(\theta) = \underline{\frac{1-\cos(2\theta)}{2}}$ ($\cos(2\theta)$ or $\sin(2\theta)$ should appear in your answer)

. arctan(-1) = $\underline{\frac{-\pi}{4}}$ **RADIANS**. (your answer should be an angle)

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 5 points.
  - For a problem with precisely two answers marked, one of which is correct, 3 points.
  - For a problem with nothing marked (i.e., left blank) 1 point.
  - All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Your Solutions							Do Not Write Below			
							points			
PROBLEM						number of solutions circled	5	3	1	0
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
							Extra Credit:			

1. $\displaystyle\int_{x=0}^{x=1} \frac{1}{x^2+1}\,dx = \arctan x\big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

2. If $u = x^2+1$, then $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$.
One can integrate the integral by trig. substitution using $x = \tan\theta$.
Note that the integrand is already in its Partial Fraction Decomposition.

3.

$$\int \frac{x}{x+9}\,dx = \int 1dx - 9\int \frac{dx}{x+9} = x - 9\ln|x+9| + C$$

$$\boxed{\frac{x}{x+9} \underset{\text{Long Division (Fake)}}{=} \frac{x+9}{x+9} - \frac{9}{x+9} = 1 - \frac{9}{x+9}}$$

Check $D_x\left[x - 9\ln|x+9|\right] = 1 - \frac{9}{x+9}$

$$= \frac{x+9}{x+9} - \frac{9}{x+9} = \frac{x}{x+9}\ . \checkmark$$

So $\displaystyle\int_0^4 \frac{x}{x+9}\,dx = \left[x - 9\ln|x+9|\right]\Big|_{x=0}^{x=4}$

$= \left[4 - 9\ln|13|\right] - \left[0 - 9\ln|9|\right]$

$= 4 - 9\ln(13) + 9\ln(9)$

4. $\displaystyle\int_{x=1}^{x=e} \ln x\,dx = (x\ln x - x)\big|_{x=1}^{x=e} = [e\ln e - e] - [1\ln 1 - 1] = [e-e] - [0-1] = 1$.

EXAMPLE 2 Evaluate $\int \ln x\,dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then $$du = \frac{1}{x}\,dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x\,dx = x\ln x - \int x\,\frac{dx}{x}$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f.

5. $\displaystyle\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x\, \sin^4 x\; dx \;=\; \left(\tfrac{1}{5}-\tfrac{1}{7}\right)-(0-0)=\tfrac{2}{35}$.

Example 4 Evaluate $\int \cos^3 x \sin^4 x\, dx$.

Solution We proceed as follows:

$$\begin{aligned}\int \cos^3 x \sin^4 x\, dx &= \int \cos^2 x \sin^4 x \cos x\, dx\\ &= \int (1-\sin^2 x)\sin^4 x \cos x\, dx.\end{aligned}$$

If we let $u = \sin x$, then $du = \cos x\, dx$ and the integral may be written

$$\begin{aligned}\int \cos^3 x \sin^4 x\, dx &= \int (1-u^2)u^4\, du\\ &= \int (u^4-u^6)\, du\\ &= \tfrac{1}{5}u^5 - \tfrac{1}{7}u^7 + C\\ &= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C.\end{aligned}$$

6. $(x^2-1)\;(x^2+1) = (x-1)\;(x+1)\;(x^2+1)$ where $x-1$ and $x+1$ are linear terms while x^2+1 is an irreducible quadratic term. Now see the partial fraction handout from class.

7. $\int_{x=-\infty}^{x=\infty} \frac{1}{1+x^2}\; dx \;=\; \pi$. From our textbook, page 506, Example 2.

HISTORICAL BIOGRAPHY
Lejeune Dirichlet
(1805–1859)

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{\infty}\frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0}\frac{dx}{1+x^2} &= \lim_{a\to-\infty}\int_{a}^{0}\frac{dx}{1+x^2}\\ &= \lim_{a\to-\infty}\tan^{-1}x\Big]_a^0\\ &= \lim_{a\to-\infty}(\tan^{-1}0-\tan^{-1}a) = 0-\left(-\frac{\pi}{2}\right)=\frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty}\frac{dx}{1+x^2} &= \lim_{b\to\infty}\int_{0}^{b}\frac{dx}{1+x^2}\\ &= \lim_{b\to\infty}\tan^{-1}x\Big]_0^b\\ &= \lim_{b\to\infty}(\tan^{-1}b-\tan^{-1}0) = \frac{\pi}{2}-0=\frac{\pi}{2}\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

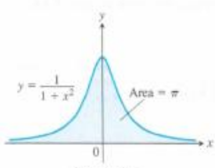

FIGURE 8.15 The area under this curve is finite (Example 2).

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15).

8. $\int_{x=-1}^{x=1} \frac{1}{x^3}\, dx$ does not exist but also does not diverge to infinity.

- $\int x^{-3}\, dx = \frac{x^{-2}}{-2} + C$

$$\int_{x=0}^{x=1} x^{-3}\, dx = \lim_{a\to 0^+} \frac{x^{-2}}{-2}\Big|_{x=a}^{x=1} = \frac{1}{2} \lim_{a\to 0^+} \left[\frac{1}{x^2}\right]_{x=1}^{x=a} =$$

$$\frac{1}{2} \lim_{x\to 0^+} \left[\frac{1}{a^2} - 1\right] = \infty.$$ Similarly, $\int_{-1}^{0} x^{-3}\, dx = -\infty$.

- $\int_{-1}^{1} x^{-3}\, dx = \int_{-1}^{0} x^{-3}\, dx + \int_{b}^{1} x^{-3}\, dx = -\infty + \infty$ so DNE.

9. From our textbook, page 512.

TABLE 8.5

b	$\int_1^b \frac{1-e^{-x}}{x}\, dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \frac{1-e^{-x}}{x}\, dx$.

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1-e^{-x}}{x}\, dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$.

10. $\int \sin x \sec x \, dx \;=\; \ln|\sec x| + C \overset{\text{or}}{=} -\ln|\cos x| + C$

$$\int (\sin x)(\sec x)\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u}$$

$u = \cos x$

$du = -\sin x\, dx$

$$= -\ln|u| + C = \boxed{-\ln|\cos x| + C}$$

$$\overset{\text{or}}{=} +\ln\left(|\cos x|\right)^{-1} + C = \ln \frac{1}{|\cos x|} + C = \boxed{\ln|\sec x| + C}$$

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11. $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx = 3\ln|x| - \ln|x+3| + 2\ln|x-1| + C$

Hint: $x^3+2x^2-3x = x(x+3)(x-1)$.

Partial Fraction Decomposition

STEP 1: Bigger Bottoms? ✓

STEP 2: Factor denominator ⇒ $x^3+2x^2-3x = x(x+3)(x-1)$ (linear, linear, linea)

STEP 3: Rewrite $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}dx = \int \frac{4x^2+13x-9}{x(x+3)(x-1)}dx$

STEP 4: Rewrite w/ factors $\frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$

STEP 5: LCD $x(x+3)(x-1)$ $\frac{4x^2+13x-9}{\cancel{x(x+3)(x-1)}} = \frac{A\cancel{x}(x+3)(x-1)}{\cancel{x}} + \frac{B\,x\cancel{(x+3)}(x-1)}{\cancel{x+3}} + \frac{C\,x(x+3)\cancel{(x-1)}}{\cancel{x-1}}$

$$4x^2+13x-9 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3)$$
$$4x^2+13x-9 = A(x^2+2x-3) + B(x^2-x) + C(x^2+3x)$$
$$4x^2+13x-9 = Ax^2+2Ax-3A+Bx^2-Bx+Cx^2+3Cx$$
$$4x^2+13x-9 = x^2(A+B+C) + x(2A-B+3C) + 1(-3A)$$

Equate coeff.

x^2: $4 = A+B+C \Rightarrow 1 = B+C$ $\quad 1 = B+2 \quad \boxed{-1=B}$

x^1: $13 = 2A-B+3C \Rightarrow 7 = -B+3C$

1: $-9 = -3A \Rightarrow \boxed{A=3}$ $\quad 8 = 4C \quad \boxed{C=2}$

Double Check

$4 = 3-1+2$ ✓

$13 = 6+1+6$ ✓

$-9 = -9$ ✓

replace constants $\int \frac{4x^2+13x-9}{x(x+3)(x-1)}dx = \int\frac{3}{x} + \int\frac{-1}{x+3} + \int\frac{2}{x-1}$

$u=x$, $du = 1dx$ $\quad u = x+3$, $du = 1dx$ $\quad u = x-1$, $du = 1dx$

$$\boxed{= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + C}$$

12. $\displaystyle\int \frac{1}{\sqrt{\ x^2+8x+25}}\ dx \ = \ \ln\left|\sqrt{x^2+8x+25}+x+4\right| + \mathrm{C}$

Example 3 Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+8x+25}}\,dx.$

Solution We complete the square for the quadratic expression as follows:

$$\begin{aligned} x^2+8x+25 &= (x^2+8x \qquad) + 25 \\ &= (x^2+8x+16)+25-16 \\ &= (x+4)^2+9. \end{aligned}$$

Hence

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{\sqrt{(x+4)^2+9}}\,dx.$$

If we next make the trigonometric substitution

$$x+4 = 3\tan\theta$$

then

$$dx = 3\sec^2\theta\,d\theta$$

$$\sqrt{(x+4)^2+9} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\tan^2\theta+1} = 3\sec\theta$$

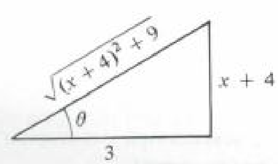
and hence

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+8x+25}}\,dx &= \int \frac{1}{3\sec\theta}\,3\sec^2\theta\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta| + C. \end{aligned}$$

In order to return to the variable x we use the triangle in Figure 10.4. This gives us

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+8x+25}}\,dx &= \ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + C \\ &= \ln\left|\sqrt{x^2+8x+25}+x+4\right| + D \end{aligned}$$

where $D = C - \ln 3$.



$x + 4 = 3\tan\theta$

Figure 10.4

13. You can think of this problem as finding an approximation of $\ln 2$ since $\int_1^2 \frac{dx}{x} = \ln 2$. Let

$$f(x) = \frac{1}{x} \qquad \text{and} \qquad [a,b] = [1,2] \qquad \text{and} \qquad n = 4\ .$$

13a. Find the Trapezidal Rule approximation T_n of $\int_a^b f(x)\, dx$ with n steps.

► Express your answer as a fraction or a sum of fractions (i.e., you do not have to perform grade-school math) and not as a decimal.

ANSWER: $T_n = \frac{1}{8}\left[1 + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) + \frac{1}{2}\right] \overset{\text{or}}{=} \frac{1}{8}\left[1 + \frac{8}{5} + \frac{4}{3} + \frac{8}{7} + \frac{1}{2}\right]$

optional table		$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$	
i	x_i	$f(x_i)$	weight
0	$1 = \frac{4}{4}$	1	1
1	$\frac{5}{4}$	$\frac{4}{5}$	2
2	$\frac{6}{4} = \frac{3}{2}$	$\frac{2}{3}$	2
3	$\frac{7}{4}$	$\frac{4}{7}$	2
4	$\frac{8}{4} = 2$	$\frac{1}{2}$	1

$$\begin{aligned} T_n &= \frac{\Delta x}{2}\left[f(x_0) + \left(2\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right] \\ &= \frac{1}{8}\left[1 + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) + \frac{1}{2}\right] \\ &= \frac{1}{8}\left[\frac{3}{2} + 2(2)\left(\frac{(2)(3)(7)+(1)(5)(7)+(2)(5)(3)}{(5)(3)(7)}\right)\right] \\ &= \frac{1}{8}\left[\frac{3}{2} + \frac{428}{105}\right] \\ &= \frac{1}{8}\left[\frac{3(105)+2(428)}{2(105)}\right] \\ &= \frac{1}{8}\left[\frac{1171}{210}\right] \\ &= \frac{1171}{1680} \end{aligned}$$

13b. Find a good upper bound for $\max_{a \le x \le b} |f''(x)|$. ANSWER: $\max_{a \le x \le b} |f''(x)| \le 2$

► Your answer should be a number.

$f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = -x^{-2} \quad \Rightarrow \quad f''(x) = 2x^{-3}$.

If $1 \le x \le 2$, then

$$|f''(x)| = \left|2x^{-3}\right| = \frac{2}{|x|^3} = \frac{2}{x^3} \le \frac{2}{1^3} = 2\ .$$

13c. The Trapezoidal Rule Error Theorem gives that $\left|T_n - \int_a^b f(x)\, dx\right| \le \frac{1}{96}$.

► Express your answer as a fraction, not a decimal.

$$\left|T_n - \int_a^b f(x)\, dx\right| \le \frac{(b-a)^3}{12n^2}\left[\max_{a \le x \le b} |f''(x)|\right] \le \frac{(2-1)^3}{12\,(4)^2}\,[2] = \frac{2}{12\,(4^2)} = \frac{1}{6\,(16)} = \frac{1}{96}.$$

13d. Find the smallest integer n so that the Trapezoidal Rule Error Theorem guarantees that $\left|T_n - \int_a^b f(x)\, dx\right| \le 10^{-3}$. ANSWER: $n = 13$.

$$\left|T_n - \int_a^b f(x)\, dx\right| \le \frac{(b-a)^3}{12n^2}\left[\max_{a \le x \le b} |f''(x)|\right] \le \frac{(2-1)^3}{12\,n^2}\,[2] = \frac{2}{12\,n^2} = \frac{1}{6\,n^2}$$

$$\frac{1}{6\,n^2} \le \frac{1}{10^3} \Leftrightarrow \frac{1000}{6} \le n^2$$

$$\frac{1000}{6} = \frac{500}{3} = 166.\overline{6} \qquad \text{and} \qquad 12^2 = 144 \qquad \text{and} \qquad 13^2 = 169$$