MARK BOX		
PROBLEM	POINTS	
0	18	
1	12	
2-11	50=10(5)	
12	10	
13	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For the Multiple Choice** problems, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
- **For problems $\geq$ 12**, to receive credit you **MUST**:
  (1) **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  (2) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (3) if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §10.7–10.10, 11.1 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0. Fill-in-the boxes.

0A. Power Series. Consider the (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n \;, \tag{1}$$

with radius of convergence $R \in [0, \infty]$.
(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)
Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

0A.1. The choices for the next 4 boxes are: AC, CC, DIVG, anthying. Here,
AC stands for: *is always absolutely convergent*
CC stands for: *is always conditionally convergent*
DIVG stands for: *is always divergent*
anything stands for: *can do anything, i.e., there are examples showing that it can AC, CC, or DIVG.*

(1) At the center $x = x_0$, the power series in (1) $\boxed{\text{AC}}$.

(2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series in (1) $\boxed{\text{AC}}$.

(3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$, the power series in (1) $\boxed{\text{DIVG}}$.

(4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series in (1) $\boxed{\text{anything}}$

0A.2. Now let $R > 0$ and fill-in the 7 boxes.
Consider the function $y = h(x)$ defined by the power series in (1).

(a) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Also, on this interval

$$h'(x) \;=\; \sum_{n=\boxed{1}}^{\infty} \boxed{n\, a_n \left(x - x_0\right)^{n-1}} \;. \tag{2}$$

What can you say about the radius of convergence of the power series in (2)?

$\boxed{\text{The power series in (2) has the same radius of convergence as the power series in (1).}}$

(b) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\, dx \;=\; \sum_{n=\boxed{0}}^{\infty} \boxed{\frac{a_n}{n+1} \left(x - x_0\right)^{n+1}} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

0B. Taylor Polynomials Series.
Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.
Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

0B.1. The formula for c_n is

$$c_n \;=\; \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

0B.2. In open form (i.e., with "..." notation and without a $\sum$-sign)

$$P_N(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N}$$

0B.3. In closed form (i.e., with a $\sum$-sign and without "..." notation)

$$P_N(x) = \boxed{\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n}$$

0B.4. In open form (i.e., with "..." notation and without a $\sum$-sign)

$$P_\infty(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots}$$

0B.5. In closed form (i.e., with a $\sum$-sign and without "..." notation)

$$P_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n}$$

0B.6. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!} (x-x_0)^{(N+1)}} \text{ for some } c \text{ between } \boxed{x} \text{ and } \boxed{x_0}.$$

0B.7. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.

1. Commonly Used Taylor Series

Here, *expansion* refers to the power series expanion that is the Maclaurin Series.

1.1. An expansion for $y = e^x$ is $\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

1.2. An expansion for $y = \cos x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

1.3. An expansion for $y = \sin x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

1.4. An expansion for $y = \frac{1}{1-x}$ is $\boxed{\sum_{n=0}^{\infty} x^n}$, which is valid precisely when $x \in \boxed{(-1, 1)}$.

1.5. An expansion for $y = \ln(1+x)$ is $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$, which is valid precisely when $x \in \boxed{(-1, 1]}$.

1.6. An expansion for $y = \arctan x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$, which is valid precisely when $x \in \boxed{[-1, 1]}$.

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 5 points.
  - For a problem with precisely two answers marked, one of which is correct, 2 points.
  - For a problem with nothing marked (i.e., left blank) 1 point.
  - All other cases, 0 points.
- Fill in the “number of solutions circled” column. (Worth a total of 1 point of extra credit.)

Your Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
							Extra Credit:			

12. Using a *Commonly Used Taylor Series*, find a power series representation center about $x_0 = -2$ for the function

$$f(x) = \frac{3}{4-5x} \ .$$

▷. State precisely for x's the representation is valid.
Put your answer in the box and justify your answer below the box.

Answer: $\sum_{n=0}^{\infty} \frac{(3)\,5^n}{14^{n+1}} (x+2)^n$ valid when $\frac{^-24}{5} < x < \frac{4}{5}$ (or equivalently, $|x - (^-2)| < \frac{14}{5}$)

A power series centered at $x_0 = -2$ has the form

$$\sum_{n=0}^{\infty} c_n \left(x - \left(^-2\right)\right)^n \overset{\text{i.e.}}{=} \sum_{n=0}^{\infty} c_n (x+2)^n$$

for some c_n's. So we want to find the special c_n's so that

$$\frac{3}{4-5x} = \sum_{n=0}^{\infty} c_n (x+2)^n \ .$$

Because of the form of f, we will use the *geometric series*

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \qquad \text{valid when } |r| < 1. \tag{GS}$$

and go for a r to be some constant times $x+2$. Now do algebraic manipulation to f , working *inside-to-outside as we do in algebra,*

$$\frac{3}{4-5x} \underset{\substack{\uparrow \\ \boxed{\text{go for } x+2 \text{ instead of } x}}}{=} \frac{3}{14-5(x+2)} \underset{\substack{\uparrow \\ \boxed{\text{want 1 where 14 is}}}}{=} \frac{3}{14\left(1 - \frac{5(x+2)}{14}\right)} \underset{\substack{\uparrow \\ \boxed{\text{want 1 on top \& } 1-r \text{ on bottom}}}}{=} \frac{3}{14} \cdot \frac{1}{1 - \frac{5(x+2)}{14}} \ . \tag{12.3}$$

So we let

$$r = \frac{5(x+2)}{14}$$

in (GS) to get

$$\frac{3}{4-5x} \overset{\text{by (12.3)}}{=} \left(\frac{3}{14}\right) \frac{1}{1 - \frac{5(x+2)}{14}} \overset{\text{by (GS)}}{=} \left(\frac{3}{14}\right) \sum_{n=0}^{\infty} \left(\frac{5(x+2)}{14}\right)^n \tag{12.4}$$

$$\overset{Ⓐ}{=} \left(\frac{3}{14}\right) \sum_{n=0}^{\infty} \frac{5^n (x+2)^n}{14^n} \overset{©}{=} \sum_{n=0}^{\infty} \left(\frac{3}{14}\right) \frac{5^n}{14^n} (x+2)^n , \tag{12.5}$$

$$\overset{Ⓐ}{=} \sum_{n=0}^{\infty} \frac{(3)\,5^n}{14^{n+1}} (x+2)^n \ . \tag{12.6}$$

The above representation is valid when the usage of (GS) is valid, i.e. when $|r| < 1$. Since

$$\left|\frac{5(x+2)}{14}\right| < 1 \quad \Leftrightarrow \quad \left|x - \left(^-2\right)\right| < \frac{14}{5} \quad \Leftrightarrow \quad ^-2 - \frac{14}{5} < x <^- 2 + \frac{14}{5}$$

$$\Leftrightarrow \quad \frac{^-10}{2} - \frac{14}{5} < x < \frac{^-10}{2} + \frac{14}{5} \quad \Leftrightarrow \quad \frac{^-24}{5} < x < \frac{4}{5} \ .$$

13. In this problem, you must show your work. Clearly explain your thought process. Using Taylor's (BIG) Remainder Theorem, show that

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \;=\; e^2 \;.$$

▷. Hint: The instruction says to use Taylor's Remainder Theorem so you cannot use the facts listed on the *Commonly Used Taylor Series* handout. (Indeed, the facts listed on the *Commonly Used Taylor Series* handout are shown by using Taylor's Remainder Theorem so think of this problem as showing one of these facts.)

▷. Hint. Consider the function $f(x) = e^x$.

Solution:

Let $f(x) = e^x$ with $f\colon \mathbb{R} \to \mathbb{R}$. Take the center $x_0 = 0$. Let's follow the notation from problem **0B**. So $f^{(n)}(x) = e^x$ and $f^{(n)}(x_0) = e^0 = 1$ for each $n \in \mathbb{N} \cup \{0\}$. So

$$e^x = P_N(x) + R_N(x)$$

and

$$e^x = P_\infty(x) \qquad \text{if and only if} \qquad \lim_{N\to\infty} |R_N(x)| = 0$$

where

$$P_N(x) \;=\; \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \;=\; \sum_{n=0}^{N} \frac{x^n}{n!}$$

and $P_\infty(x) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and Taylor's Remainder Theorem tells us that

$$\text{for some } c \text{ between } x \text{ and } x_0\text{:} \qquad R_N(x) \;=\; \frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)} \;=\; \frac{e^c x^{N+1}}{(N+1)!}\;. \tag{13.7}$$

Taking $x = 2$ in (13.7) gives that for some c between 2 and 0:

$$|R_N(2)| \;=\; \frac{e^c\;|x|^{N+1}}{(N+1)!} \;=\; \frac{e^c 2^{N+1}}{(N+1)!} \;\stackrel{0\le c\le 2}{\le}\; \frac{e^2\,2^{N+1}}{(N+1)!} \tag{13.8}$$

Next we want to show that the (good) upper bound we found in (13.8) tends to zero as $N \to \infty$. So we want to show that $\lim_{N\to\infty} \frac{e^2\,2^{N+1}}{(N+1)!} = 0$. (Sometimes this step is easy but in this example we will have to use a little trick (tool) that sometimes works … here we go). Let $a_N = \frac{e^2\,2^{N+1}}{(N+1)!}$. (To show that $\lim_N a_N = 0$, we will actually show something stronger, namely $\sum a_N$ converges.) The Ratio Test tells us that the series $\sum_{n=0}^{\infty} \dfrac{e^2\,2^{n+1}}{(n+1)!}$ is (absolutely) convergent since applying the Ratio Test we get

$$\rho = \lim_{N\to\infty} \left| \frac{a_{N+1}}{a_N} \right| = \lim_{N\to\infty} \frac{e^2\,2^{N+2}}{(N+2)!} \cdot \frac{(N+1)!}{e^2\,2^{N+1}} = \lim_{N\to\infty} \frac{2}{N+2} = 0.$$

The n^{th} term test for divergence gives that if the series $\sum_n a_n$ converges, then the limit of the sequence $\{a_n\}_n$ is 0, i.e. $\lim_{n\to\infty} a_n = 0$. So

$$\lim_{N\to\infty} \frac{e^2\,2^{N+1}}{(N+1)!} = 0\;. \tag{13.9}$$

So

$$0 \le |R_N(2)| \;\stackrel{\text{by (13.8)}}{\le}\; \frac{e^2\,2^{N+1}}{(N+1)!} \xrightarrow{\text{as } N\to\infty,\text{ by (13.9)}} 0.$$

The Squeeze/Sandwich Theorem gives that $\lim_{N\to\infty} |R_N(2)| = 0$.
So $e^2 = P_\infty(2)$, i.e.,

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}\;.$$

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are <u>not</u> collected.

2. Find the 2nd order Taylor polynomial for $f(x) = \sqrt[3]{x}$ about the center $x_0 = 8$.

2soln. The 2nd order Taylor polynomial $P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$. Using the below Helpful Table we get $P_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{9(2^5)}(x-8)^2$.

Helpful Table			
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(8)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(8)}{n!}$
0	$x^{\frac{1}{3}}$	$8^{1/3} = 2$	$\frac{2}{0!} = 2$
1	$\frac{1}{3}x^{\frac{-2}{3}}$	$\frac{1}{3}\left(8^{1/3}\right)^{-2} = \frac{1}{3}\frac{1}{2^2} = \frac{1}{12}$	$\frac{\left(\frac{1}{12}\right)}{1!} = \frac{1}{12}$
2	$\frac{-2}{9}x^{\frac{-5}{3}}$	$\frac{-2}{9}\left(8^{1/3}\right)^{-5} = \frac{-2}{9}\frac{1}{2^5} = \frac{-2}{9(2^5)}$	$\left(\frac{1}{2!}\right)\cdot\left(\frac{-2}{9(2^5)}\right) = \frac{-1}{9(2^5)}$

3. Using a *Commonly Used Taylor Series*, find the Tayor series for $f(x) = x\cos(4x)$ about $x_0 = 0$.

3soln. The *Commonly Used Taylor Series* $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ is valid for $-\infty < x < \infty$. So

$$x\cos(4x) \overset{(*)}{=} x\sum_{n=0}^{\infty}(-1)^n \frac{(4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x\,(4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{4^{2n}\,x^{2n+1}}{(2n)!}\,.$$

This expansion is valid whenever (*) is valid, so when $-\infty < 4x < \infty$, so when $-\infty < x < \infty$.

4. Find the Taylor series for $f(x) = (1-5x)^{-3}$ about the center $x_0 = 0$.

4soln. The *Geometric Series* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is valid when $-1 < x < 1$. Differentiating (see **0A.2**):

$$\begin{aligned}
(1-x)^{-1} &= \sum_{n=0}^{\infty} x^n \\
D_x\left((1-x)^{-1}\right) = (1-x)^{-2} &= \sum_{n=1}^{\infty} n\,x^{n-1} \\
D_x\left((1-x)^{-2}\right) = 2\,(1-x)^{-3} &= \sum_{n=2}^{\infty} n\,(n-1)\,x^{n-2}\,.
\end{aligned}$$

So

$$(1-x)^{-3} = \sum_{n=2}^{\infty} \frac{n\,(n-1)}{2}\,x^{n-2}$$

which is valid when $-1 < x < 1$. So

$$\begin{aligned}(1-(5x))^{-3} &= \sum_{n=2}^{\infty} \frac{n\,(n-1)}{2}\,(5x)^{n-2} = \sum_{n=2}^{\infty} \frac{n\,(n-1)\,5^{n-2}}{2}\,x^{n-2} \\ &= \frac{2\,(1)\,5^0}{2}\,x^0 + \frac{3\,(2)\,5^1}{2}\,x^1 + \frac{4\,(3)\,5^2}{2}\,x^2 + \ldots \\ &\underset{n-2=k}{\overset{\text{or let}}{=}} \sum_{k=0}^{\infty} \frac{(k+2)\,(k+1)\,5^k}{2}\,x^k \\ &= \sum_{n=0}^{\infty} \frac{(n+2)\,(n+1)\,5^n}{2}\,x^n\end{aligned}$$

which is valid when $-1 < 5x < 1$, so when $-\frac{1}{5} < x < \frac{1}{5}$.

5. Using a *Commonly Used Taylor Series*, evaluate $\int \tan^{-1}(t^2)\,dt$ as a power series.

5soln. The *Commonly Used Taylor Series* $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ is valid when $-1 \le x \le 1$. So

$$\tan^{-1}\left(t^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(t^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{2n+1}\ ,$$

which is valid when $-1 \le t^2 \le 1$, so when $-1 \le t \le 1$. By integrating the above power expansion (see Problem **0A**),

$$\int \tan^{-1}\left(t^2\right)\,dt = \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{2n+1}\right] dt = \sum_{n=0}^{\infty} \left[\int \frac{(-1)^n t^{4n+2}}{2n+1}\,dt\right] = C + \sum_{n=0}^{\infty} \left[\frac{(-1)^n t^{4n+3}}{(2n+1)\,(4n+3)}\,dt\right]\ ,$$

which is valid when (might loose endpoints) $-1 < t < 1$.

6. Find a power series representation of the function $y = f(t)$ where

$$f(t) = \int_0^t \frac{1}{1+x^7}\,dx$$

and say for which values of t it is valid.

6soln. The *Geometric Series* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is valid when $-1 < x < 1$. So (using Probelm **0A**),

$$\begin{aligned} f(t) &= \int_0^t \frac{1}{1+x^7}\,dx = \int_0^t \frac{1}{1-(-x^7)}\,dx \overset{\text{GS}}{=} \int_0^t \left[\sum_{n=0}^{\infty} \left(-1\,x^7\right)^n\right] dx = \int_0^t \left[\sum_{n=0}^{\infty} (-1)^n\,x^{7n}\right] dx \\ &= \sum_{n=0}^{\infty} \left[\int_0^t (-1)^n\,x^{7n}\,dx\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1}\,x^{7n+1}\Big|_{x=0}^{x=t} \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1}\,t^{7n+1}\ .\end{aligned}$$

The step at GS which is valid when $-1 < x^7 < 1$, so when $-1 < x < 1$, i.e. when $x \in [-1, 1]$. So the expansion is valid when the interval between 0 and t is in $(-1, 1)$ (might loose the endpoints), so when $-1 < t < 1$.

7. Suppose that the power series $\sum_{n=1}^{\infty} c_n (x-10)^n$ has interval of convergence $(1, 19)$. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} c_n x^{2n}$?

7soln.

$$\sum_{n=1}^{\infty} c_n x^{2n} \text{ converges } \Leftrightarrow \sum_{n=1}^{\infty} c_n \left(x^2\right)^n \text{ converges } \Leftrightarrow \sum_{n=1}^{\infty} c_n \left[\left(x^2+10\right)-10\right]^n \text{ converges}$$

$$\overset{\text{given}}{\Leftrightarrow} 1 < x^2+10 < 19 \quad \Leftrightarrow \quad -9 < x^2 < 9 \quad \Leftrightarrow \quad -3 < x < 3 \, .$$

8. Suppose that the interval of convergence of the series $\sum_{n=1}^{\infty} c_n(x-x_0)^n$ is $(x_0-R, x_0+R]$. What can be said about the series at x_0+R?

8soln.

The assumption is that $\sum_{k=0}^{\infty} c_k R^k$ is convergent and $\sum_{k=0}^{\infty} c_k(-R)^k$ is divergent. Suppose that $\sum_{k=0}^{\infty} c_k R^k$ is absolutely convergent then $\sum_{k=0}^{\infty} c_k(-R)^k$ is also absolutely convergent and hence convergent because $|c_k R^k| = |c_k(-R)^k|$, which contradicts the assumption that $\sum_{k=0}^{\infty} c_k(-R)^k$ is divergent so $\sum_{k=0}^{\infty} c_k R^k$ must be conditionally convergent.

9. Consider the function $f(x) = (7-x)^{-2}$ about the center $x_0 = 5$ over the interval $J = (4,6)$. For each $x \in J$, find a good upper bound for the absolute value of the N^{th}-order Taylor remainder term $R_N(x)$. Your answer can have an N and x in it but it cannot have an: x_0,c.

9soln. See the Take-home part of Fall 2009 Exam 2, part h, at

http://people.math.sc.edu/girardi/m142/exam/09Fe2Taylorsoln.pdf

10. Find the 10^{th}-order Taylor polynomial $y = P_{10}(x)$, center about $x_0 = 17$, of the function

$$f(x) = 5 + 6x^7$$

10soln. Note $f^{(n)}(x) = 0$ for each $n \geq 8$ and $x \in \mathbb{R}$. Let's following notation from Probelm **0B**. If $N \geq 7$, then $f^{(N+1)}(c) = 0$ for any $c \in \mathbb{R}$ and so

$$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} (x-17)^{N+1} \right| = \frac{0}{(N+1)!} |x-17|^{N+1} = 0 \, ,$$

and so $P_N(x) = f(x)$. So $p_{10}(x) = f(x)$.

11. A parametrization of a circle with center at $(0,0)$ and raduis 1, which is traced out twice in the clockwise direction is

11soln. $x(t) = \cos t$ and $y(t) = -\sin t$ for $0 \leq t \leq 4\pi$. Note $[x(t)]^2 + [y(t)]^2 = 1$ so the puffo is running around a circle with radius 1 and center $(0,0)$. The the negative on the y makes the tracing go clockwise while $0 \leq t \leq 4\pi$ taces the circle twice.