MARK BOX		
PROBLEM	POINTS	
0	10	
1	10	
2-13	60=12x5	
14	10	
15	10	
%	100	

HAND IN PART

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  - (1) HAND IN PART. Hand in only this part.
  - (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For TrueFalse/MultipleChoice problems 1–13**, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
- **For problems > 13**, to receive credit you **MUST**:
  - (1) **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  - (2) if a line/box is provided, then:
    — show you work BELOW the line/box
    — put your answer on/in the line/box
  - (3) if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.7–8.8, 10.1–10.7 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________________

0. Fill-in-the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0.1. Geometric Series. Fill in the boxes with the proper range of $r \in \mathbb{R}$.

- The series $\sum r^n$ converges if and only if r satisfies $\boxed{|r| < 1}$.

0.2. p-series. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if $\boxed{p > 1}$.

0.3. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If $\boxed{0 \le a_n \le c_n}$ (only $a_n \le c_n$ is also ok b/c given $a_n \ge 0$) when $n \ge 17$ and $\boxed{\sum c_n \text{ converges}}$, then $\sum a_n$ converges.
- If $\boxed{0 \le d_n \le a_n}$ (need $0 \le d_n$ part here) when $n \ge 17$ and $\boxed{\sum d_n \text{ diverges}}$, then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.4. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then $\boxed{[\sum b_n \text{ converges} \iff \sum a_n \text{ converges}]}$
- If $L = 0$, then $\boxed{[\sum b_n \text{ converges} \implies \sum a_n \text{ converges}]}$.
- If $L = \infty$, then $\boxed{[\sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}]}$.

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.5. Helpful Intuition Fill in the 3 boxes using: e^x, $\ln x$, x^q. Use each once, and only once.

Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \ge N_q$ then

$$\boxed{\ln x} \le \boxed{x^q} \le \boxed{e^x} .$$

1. Circle T if the statement is TRUE. Circle F if the statement if FALSE. To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true. Scoring: 2 pts for correct answer, 0 pts for an incorrect answer, 1 pt for a blank answer (indicated by a circled B).

T	F	B	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
T	F	B	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
T	F	B	If $a_n \ge 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ is either absolutely convergent or divergent.
T	F	B	If $\sum |a_n|$ converges, then $\sum a_n$ converges.
T	F	B	If $\sum (a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 5 points.
  - For a problem with precisely two answers marked, one of which is correct, 3 points.
  - For a problem with nothing marked (i.e., left blank) 1 point.
  - All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Your Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
13	13a	13b	13c	13d	13e					
							Extra Credit:			

Fall 2016 Exam 2

14. Geometric Series. (On this page, you should do basic algebra but you do NOT have to do any grade-school arithmetic (eg, you can leave $(\frac{17}{18})^{171}$ as just that.) Let, for $N \geq 51$,

$$s_N = \sum_{n=51}^{N} 2\frac{3^{n+1}}{5^n}.$$

14a. Do some algebra to write s_N as $\sum_{n=51}^{N} c\, r^n$ for an appropriate constant c and ratio r.

$$\boxed{s_N = \sum_{n=51}^{N} 6\left(\frac{3}{5}\right)^n}$$

$$2\frac{3^{n+1}}{5^n} = 2 \cdot 3 \cdot \frac{3^n}{5^n} = 6\left(\frac{3}{5}\right)^n$$

$c = 6$

$r = \frac{3}{5}$

14b. Using the method from class (rather than some formula), find an expression for s_N in closed form (i.e. without a summation $\sum$ sign nor some dots ....).

$$\boxed{s_N = \frac{6\left(\frac{3}{5}\right)^{51} - 6\left(\frac{3}{5}\right)^{N+1}}{1-\frac{3}{5}} = 15\left(\left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1}\right)}$$

$$S_N = 6\left(\frac{3}{5}\right)^{51} + 6\left(\frac{3}{5}\right)^{52} + 6\left(\frac{3}{5}\right)^{53} + \dots 6\left(\frac{3}{5}\right)^{n}$$

$$-rS_N = 6\left(\frac{3}{5}\right)^{52} + 6\left(\frac{3}{5}\right)^{53} + 6\left(\frac{3}{5}\right)^{54} + 6\left(\frac{3}{5}\right)^{N+1}$$

$$S_N - rS_N = 6\left(\frac{3}{5}\right)^{51} - 6\left(\frac{3}{5}\right)^{N+1}$$

$S_N - rS_N$

$S_N(1-r) = \sim$

$S_N = \frac{\sim}{1-r}$

$$S_N = \frac{6\left(\frac{3}{5}\right)^{51} - 6\left(\frac{3}{5}\right)^{N+1}}{1-r} = \frac{6\left(\frac{3}{5}\right)^{51} - 6\left(\frac{3}{5}\right)^{N+1}}{1-\frac{3}{5}}$$

14c. Does $\sum_{n=51}^{\infty} 2\frac{3^{n+1}}{5^n}$ converge or diverge? If it converges, find its sum. Justify your answer.

$\boxed{\sum_{n=51}^{\infty} 2\frac{3^{n+1}}{5^n} \quad \text{Converges to } 15\left(\frac{3}{5}\right)^{51}}$

$$\lim_{n\to\infty} 15\left(\left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{n+1}\right) = 15\left(\left(\frac{3}{5}\right)^{51} - 0\right) = 15\left(\frac{3}{5}\right)^{51}$$

$\left(\frac{3}{5}\right)^{n+1}$ as $n \to \infty$ $= 0$

geometric series when $r = \frac{3}{5} < 1$

Prof. Girardi Page 4 of 9

Fall 2016 Exam 2

15. Check the correct box and then indicate your reasoning below. **SHOW ALL YOUR WORK.** Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.23}}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

do not want a Σ sign here

Use DCT ✓ $\cancel{\sum_{n=1}}$ $\frac{\ln(n)}{n^{1.23}} \leq \frac{n^q}{n^{1.23}} = \frac{1}{n^{1.23-q}}$

$1.23 - q > 1$

$-q > -.23$

$q < .23$

let $q = .10$

w/ $q = .10$ $\frac{1}{n^{1.23-.10}} = \frac{1}{n^{1.13}} = b_n$

$\frac{\ln(n)}{n^{1.23}} = a_n$

$0 \leq a_n \leq b_n$

$0 \leq \frac{\ln(n)}{n^{1.23}} \leq \frac{1}{n^{1.13}}$

therefore this a conv. series

this is a convergent P-series, and bounded Σ above Σa_n

$\therefore$

beautiful!

good! Since $\Sigma |a_n| = \Sigma a_n$, Σa_n is a absolutely convergent series

Prof. Girardi Page 5 of 9

Fall 2016 Exam 2

15. Check the correct box and then indicate your reasoning below. **SHOW ALL YOUR WORK.** Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$\sum a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^{1.23}}$

- [x] absolutely convergent
- [ ] conditionally convergent — $a_n \geq 0$ for all $n \in \mathbb{N}$; cannot be COND. CONV.
- [ ] divergent

* useful intuition!

$\ln n \leq n^q \leq e^n$; for $q > 0$ and some value of n large enough.

$\frac{\ln n}{n^{1.23}} \leq \frac{n^q}{n^{1.23}}$; $\frac{n^q}{n^{1.23}} = \frac{1}{n^{1.23-q}}$

$\boxed{\text{USE DIRECT COMPARISON TEST}}$

- $\sum a_n$ is bounded above by $\sum \frac{1}{n^{1.23-q}}$;
- to converge, $\sum \frac{1}{n^{1.23-q}}$, a p-series, $1.23 - q = p > 1$

$1.23 - q > 1 \Rightarrow -q > -0.23 \Rightarrow q < 0.23$

- $q = 0.22$

Since $\frac{\ln n}{n^{1.23}} \leq \frac{1}{n^{1.01}}$

and $\sum \frac{1}{n^{1.01}}$ is a p-series w/ $p > 1$

it follows that $\sum \frac{\ln n}{n^{1.23}}$ also converges.

nice!

2soln. $\int_{x=-\infty}^{x=\infty} \frac{1}{1+x^2}\,dx = \pi$. From our textbook, page 506, Example 2.

HISTORICAL BIOGRAPHY
Lejeune Dirichlet
(1805–1859)

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} \\ &= \lim_{a\to-\infty} \tan^{-1} x \Big]_a^0 \\ &= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\ &= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

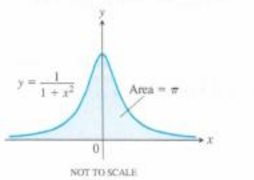


FIGURE 8.15 The area under this curve is finite (Example 2).

Since $1/(1 + x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15).

3soln.

$$\int \frac{-2}{(x-11)^3}\,dx = \int -2(x-11)^{-3}\,dx = \frac{-2(x-11)^{-2}}{-2} + C = \frac{1}{(x-11)^2} + C$$

10 $a \to 11 \leftarrow b$ 12

$$\int_{10}^{12} \frac{-2}{(x-11)^3}\,dx = \left[\lim_{a\to 11^-} \int_{10}^{a} \frac{-2}{(x-11)^3}\,dx\right] + \left[\lim_{b\to 11^+} \int_{b}^{12} \frac{-2}{(x-11)^3}\,dx\right]$$

$$= \left[\lim_{a\to 11^-} \frac{1}{(x-11)^2}\Big|_{x=10}^{x=a}\right] + \left[\lim_{b\to 11^+} \frac{1}{(x-11)^2}\Big|_{x=b}^{x=12}\right]$$

$$= \left[\lim_{a\to 11^-} \left(\frac{1}{(a-11)^2} - 1\right)\right] + \left[\lim_{b\to 11^+} \left(1 - \frac{1}{(b-11)^2}\right)\right]$$

$$= \left[\lim_{a\to 11^-} \frac{1}{(a-11)^2}\right] - 1 + 1 - \left[\lim_{b\to 11^+} \frac{1}{(b-11)^2}\right]$$

$$= \underbrace{\left[\lim_{a\to 11^-} \frac{1}{(a-11)^2}\right]}_{\infty} - \underbrace{\left[\lim_{b\to 11^+} \frac{1}{(b-11)^2}\right]}_{\infty}$$

$$\boxed{\infty - \infty}$$

→ indeterminate form, DNE.

4soln. From our textbook, page 512.

TABLE 8.5

b	$\int_1^b \frac{1-e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

EXAMPLE 9 Investigate the convergence of $\displaystyle\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$.

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \leq g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\displaystyle\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$ diverges because $\displaystyle\int_1^{\infty} \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$.

5soln.

$$\frac{\sqrt{25n^3 + 4n^2 + n - 5}}{7n^{3/2} + 6n - 1} = \frac{\sqrt{25 + \frac{4}{n} + \frac{1}{n^2} - \frac{5}{n^3}}}{7 + \frac{6}{n^{1/2}} - \frac{1}{n^{3/2}}} \xrightarrow{n\to\infty} \frac{\sqrt{25}}{7} = \frac{5}{7}.$$

$\boxed{\div \text{ num \& dem. by } n^{3/2} = \sqrt{n^3}}$

6soln.

Want $\displaystyle\sum_{n=2}^{\infty} r^n = \frac{1}{12}$. We know that we <u>must</u> have $|r| < 1$ (geom. series)

Let $S_N = \displaystyle\sum_{n=2}^{N} r^n$, So

$$1\,S_N = r^2 + r^3 + r^4 + \cdots + r^N$$

stract $\quad r\,S_N = \qquad\quad r^3 + r^4 + \cdots + r^N + r^{N+1}$

$$(1-r)\,S_N = r^2 - r^{N+1}$$

$$S_N \underset{=}{\overset{r\neq 1}{=}} \frac{r^2 - r^{N+1}}{1-r} \xrightarrow[\text{if } |r|<1]{N\to\infty} \frac{r^2}{1-r}$$

So want $r \in \mathbb{R}$ so that $|r| < 1$ and $\frac{r^2}{1-r} = \frac{1}{12}$

$$\frac{r^2}{1-r} = \frac{1}{12} \iff 12r^2 = 1 - r \iff 12r^2 + r - 1 = 0$$

$$\iff r = \frac{-1 \pm \sqrt{1^2 - 4(12)(-1)}}{2(12)} = \frac{-1 \pm \sqrt{1+48}}{2\cdot 12} = \frac{-1 \pm 7}{2\cdot 12}$$

$$= \begin{cases} \dfrac{-1-7}{2\cdot 12} = \dfrac{-8}{2\cdot 12} = -\dfrac{2\cdot 4}{2\cdot 4\cdot 3} = -\dfrac{1}{3} \\[2ex] \dfrac{-1+7}{2\cdot 12} = \dfrac{6}{2\cdot 6\cdot 2} = \dfrac{1}{4} \end{cases}$$

7soln.

Know (A) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p=1$, $p \le 1$) (or, also, harmonic seri

(B) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (just apply the AST).

Now, look at the choices:

8soln. $\frac{1}{\sqrt{(n+2)(n+7)}} \overset{n \text{ big}}{\sim} \frac{1}{\sqrt{(n)(n)}} = \frac{1}{n}$. So let $b_n = \frac{1}{n}$ and $a_n = \frac{(-1)^n}{\sqrt{(n+2)(n+7)}}$. Then

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{(n+2)(n+7)}} = \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{(n+2)(n+7)}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{(n+2)(n+7)}} = \sqrt{1} = 1$$

Since $0 < \lim_{n\to\infty} \frac{|a_n|}{b_n} < \infty$, by the LCT, $\sum b_n$ and $\sum |a_n|$ *do the same thing* and we know that $\sum b_n$ is the harmonic series so $\sum b_n$ is diverges. So $\sum |a_n|$ diverges.

Now let $u_n = \frac{1}{\sqrt{(n+2)(n+7)}}$. Since $0 \le u_n \searrow 0$, by the AST, $\sum (-1)^n u_n$ converges.

Now look at the choices.

9soln.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{[3(n+1)]!} \cdot \frac{(3n)!}{n!} = \frac{(n!)(n+1)}{(n!)} \cdot \frac{(3n)!}{[(3n)!](3n+1)(3n+2)(3n+3)}$$

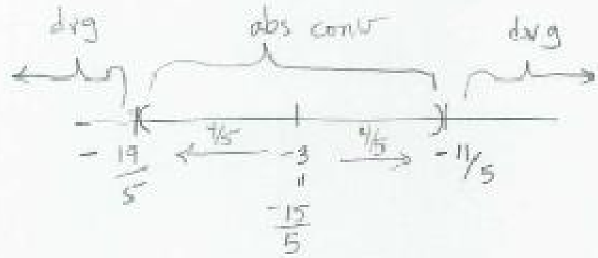
$$= \frac{n+1}{(3n+1)(3n+2)(3n+3)} \xrightarrow{n\to\infty} 0.$$

10soln.

$\sum \frac{(5x+15)^n}{4^n} = \sum \frac{[5(x+3)]^n}{4^n} = \sum \left(\frac{5}{4}\right)^n (x - -3)^n \; . \Rightarrow$ center =

$\left[\left| \frac{(5x+15)^n}{4^n} \right| \right]^{1/n} = \left| \frac{5x+15}{4} \right| = \frac{5}{4}|x+3| \xrightarrow{n\to\infty} \frac{5}{4}|x+3|$

$\frac{5}{4}|x+3| < 1 \iff |x+3| < \frac{4}{5} \Rightarrow$ rad. of conv. is $\frac{4}{5}$



Check endpts

$x = -11/5$: $\sum \frac{(5x+15)^n}{4^n} = \sum \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1$ <u>dvg</u> (to ∞)

$x = -19/5$: $\sum \frac{(5x+15)^n}{4^n} = \sum \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n$ dvg (osc.)

11soln. Let

$$a_n = \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5)}}$$

For n sufficiently big,

$$a_n = \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5)}} \overset{\text{when } n \text{ is big}}{\approx} \frac{n}{\sqrt{(n)\,(n)\,(n)\,(n)\,(n)}} = \frac{n^1}{n^{5/2}} = \frac{1}{n^{3/2}}\,.$$

So we let $b_n = \left(\frac{1}{n}\right)^{3/2}$ and compute

$$\begin{aligned}
\frac{a_n}{b_n} &= \frac{n}{\sqrt{(n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5)}}\,\frac{n^{3/2}}{1} = \frac{n^{5/2}}{[\,(n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5)\,]^{1/2}} \\
&= \left[\frac{n^5}{(n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5)}\right]^{1/2} = \left[\frac{n}{(n+1)}\,\frac{n}{(n+2)}\,\frac{n}{(n+3)}\,\frac{n}{(n+4)}\,\frac{n}{(n+5)}\right]^{1/2} \\
&\xrightarrow{n\to\infty} [\,(1)\;(1)\;(1)\;(1)\;(1)\,]^{1/2} = 1\,.
\end{aligned}$$

Since $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, the LCT says the $\sum a_n$ and $\sum b_n$ do the same thing. Since $\sum b_n$ is a p-series with $p = \frac{3}{2} > 1$, the $\sum b_n$ converges. So the $\sum a_n$ converges.

12soln. Basically, apply the integral test. Note that the computation of

$$\int \frac{1}{x \ln x}\, dx \;=\; \ln|\ln|u|| \;+\; C \;,$$

as seen by using a u-du subsitution with $u = \ln x$. Let's see what happens if we try the ratio test. Let $a_n = \frac{1}{n \ln n}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \;=\; \lim_{n\to\infty} \frac{1}{(n+1)\,\ln(n+1)}\,\frac{n\,\ln n}{1} \;=\; \lim_{n\to\infty} \frac{n}{n+1}\,\frac{\ln n}{\ln(n+1)} \;.$$

By L'Hopital's Rule,

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \;=\; \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} \;=\; \lim_{n\to\infty} \frac{n+1}{n} \;=\; 1 \;.$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the Ratio Test is inconclusive.

13soln.

SECTION 11.2 SERIES |||| 691

EXAMPLE 6 Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

(see Section 7.4). Thus we have

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

and so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

□