HAND IN PART

MARK BOX		
PROBLEM	POINTS	
1-20	100	
%	100	

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- **For the Multiple Choice** problems, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
- The MARK BOX above indicates the problems along with their points. Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1–8.5, 8.7–8.8, 10.1–10.10, 11.1–11.5 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

* Indicate (by circling) directly in the table below your solution to each problem.
* You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 5 points.
  - For a problem with precisely two answers marked, one of which is correct, 2 points.
  - For a problem with nothing marked (i.e., left blank) 1 point.
  - All other cases, 0 points.
* Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Table for Your Muliple Choice Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	1c	1d	**1e**					
2	2a	**2b**	2c	2d	2e					
3	3a	3b	**3c**	3d	3e					
4	4a	4b	4c	4d	**4e**					
5	5a	5b	5c	**5d**	5e					
6	6a	6b	6c	**6d**	6e					
7	**7a**	7b	7c	7d	7e					
8	8a	8b	8c	**8d**	8e					
9	9a	**9b**	9c	9d	9e					
10	10a	10b	10c	**10d**	10e					
11	11a	11b	**11c**	11d	11e					
12	12a	**12b**	12c	12d	12e					
13	**13a**	13b	13c	13d	13e					
14	14a	14b	**14c**	14d	14e					
15	15a	15b	15c	**15d**	15e					
16	16a	16b	**16c**	16d	16e					
17	17a	17b	17c	17d	**17e**					
18	18a	18b	**18c**	18d	18e					
19	19a	19b	**19c**	19d	19e					
20	**20a**	**20b**	**20c**	**20d**	**20e**	17				

5	2	1	0
Extra Credit:			

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

- Hint. For a typical (i.e. not improper) definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.

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1. Evaluate the integral

$$\int_{x=0}^{x=\frac{\pi}{2}} \sin^2 x \, \cos x \, dx$$

1soln. First do indefinite integral. Let $u = \sin x$ so $du = \cos x dx$. So

$$\int \sin^2 x \, \cos x \, dx \;=\; \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C \; .$$

Next check indefinite integral:

$$D_x \left(\frac{\sin^3 x}{3} \right) = \frac{1}{3} \, D_x \left((\sin x)^3 \right) = \frac{1}{3} \left(3 \sin^{3-1} x \right) D_x \sin x = \sin^2 x \cos x \quad \boxed{\checkmark}$$

So

$$\int_{x=0}^{x=\frac{\pi}{2}} \sin^2 x \, \cos x \, dx \;=\; \frac{\sin^3 x}{3} \Big|_{x=0}^{x=\frac{\pi}{2}} \;=\; \frac{1}{3} \left(\sin^3 \frac{\pi}{2} - \sin^3 0 \right) \;=\; \frac{1}{3} \left(1^3 - 0^3 \right) = \frac{1}{3} \; .$$

2. Evaluate the integral

$$\int_{x=0}^{x=\frac{\pi}{4}} \sin^2 x \, dx$$

2soln. First do indefinite integral.

Solution Here we make use of half-angle identities.

$$\begin{aligned} \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \int dx - \frac{1}{4} \int (\cos 2x)(2 \, dx) \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \end{aligned}$$

Next check indefinite integral:

$$D_x \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) = \frac{1}{2} - \frac{D_x (\sin 2x)}{4} = \frac{1}{2} - \frac{(2 \cos 2x)}{4} = \frac{1 - \cos 2x}{2} = \sin^2 x \quad \boxed{\checkmark} \; .$$

So

$$\int_{x=0}^{x=\frac{\pi}{4}} \sin^2 x\,dx = \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\Big|_{x=0}^{x=\frac{\pi}{4}} = \left(\frac{\pi}{8} - \frac{\sin\frac{\pi}{2}}{4}\right) - \left(\frac{0}{2} - \frac{\sin 0}{4}\right) = \left(\frac{\pi}{8} - \frac{1}{4}\right) - \left(\frac{0}{2} - \frac{0}{4}\right)$$
$$= \frac{\pi}{8} - \frac{1}{4}.$$

3. Evaluate the integral

$$\int_{x=0}^{x=\sqrt{\frac{\pi}{2}}} x\,\sin\left(x^2\right)\,dx$$

3soln. First do indefinite integral. Simple u-du substitution with $u = x^2$ and so $du = 2xdx$.

$$\int x\,\sin\left(x^2\right)\,dx = \tfrac{1}{2}\int \sin\left(x^2\right)\boxed{2x\,dx} = \tfrac{1}{2}\int \sin u\,du = \tfrac{-\cos u}{2} + C = \tfrac{-\cos x^2}{2} + C\ .$$

Next check indefinite integral:

$$D_x\left(\tfrac{-\cos x^2}{2}\right) = \tfrac{-1}{2}D_X\left(\cos\left(x^2\right)\right) = \tfrac{-1}{2}\left({}^-\sin x^2\right)(2x) = x\sin x^2 \quad \boxed{\checkmark}\,.$$

So

$$\int_{x=0}^{x=\sqrt{\frac{\pi}{2}}} x\,\sin\left(x^2\right)\,dx = \frac{-\cos\left(x^2\right)}{2}\Big|_{x=0}^{x=\sqrt{\frac{\pi}{2}}} = \left(\frac{-\cos\frac{\pi}{2}}{2}\right) - \left(\frac{-\cos 0}{2}\right) = (0) - \left(\frac{-1}{2}\right) = \frac{1}{2}\ .$$

4. Evaluate the integral

$$\int_0^{\pi} e^{5x}\cos 3x\,dx$$

4soln. To do the (indefinite) integral, we will use two integration by parts and the *bring to the other side* idea. For the two integration by parts, put the expontential function with either the u's both times or the dv's both times.

Way # 1

For this way, for each integration by parts, we let the u involve the expontenial function.

$$u_1 = e^{5x} \qquad dv_1 = \cos 3x\,dx$$
$$du_1 = 5e^{5x}\,dx \qquad v_1 = \frac{1}{3}\sin 3x\ .$$

So by integration by parts

$$\int e^{5x}\,\cos 3x\,dx = \frac{1}{3}e^{5x}\sin 3x - \frac{5}{3}\int e^{5x}\sin 3x\,dx\ .$$

Now let

$$u_2 = e^{5x} \qquad dv_2 = \sin 3x\, dx$$
$$du_2 = 5e^{5x}\, dx \qquad v_2 = \frac{-1}{3}\cos 3x\ .$$

to get

$$\begin{aligned}\int e^{5x}\cos 3x\ dx &= \frac{1}{3}e^{5x}\sin 3x - \frac{5}{3}\left[\frac{-1}{3}e^{5x}\cos 3x - \frac{-5}{3}\int e^{5x}\cos 3x\, dx\right] \\ &= \frac{1}{3}e^{5x}\sin 3x + \frac{5}{3^2}e^{5x}\cos 3x - \frac{5^2}{3^2}\int e^{5x}\cos 3x\, dx\ .\end{aligned}$$

Now solving for $\int e^{5x}\cos 3x\, dx$ (use the *bring to the other side* idea) we get

$$\left[1+\frac{5^2}{3^2}\right]\int e^{5x}\cos 3x\ dx = \frac{1}{3}e^{5x}\sin 3x + \frac{5}{3^2}e^{5x}\cos 3x + K$$

and so

$$\begin{aligned}\int e^{5x}\cos 3x\ dx &= \left[\frac{3^2}{34}\right]\left(\frac{1}{3}e^{5x}\sin 3x + \frac{5}{3^2}e^{5x}\cos 3x + K\right) \\ &= \frac{3}{34}\ e^{5x}\sin 3x + \frac{5}{34}\ e^{5x}\cos 3x + \left[\frac{K3^2}{34}\right] \\ &= \frac{e^{5x}}{34}\ (3\sin 3x + 5\cos 3x) + \left[\frac{K3^2}{34}\right]\ .\end{aligned}$$

Thus

$$\int e^{5x}\cos 3x\ dx = \frac{e^{5x}}{34}\ (5\cos 3x + 3\sin 3x) + C\ .$$

Way # 2

For this way, for each integration by parts, we let the dv involve the expontenial function.

$$u_1 = \cos 3x \qquad dv_1 = e^{5x}dx$$
$$du_1 = -3\sin 3x\, dx \qquad v_1 = \frac{1}{5}\, e^{5x}\ .$$

So, by integration by parts

$$\int e^{5x}\cos 3x\ dx = \frac{1}{5}e^{5x}\cos 3x - \frac{-3}{5}\int e^{5x}\sin 3x\, dx\ .$$

Now let

$$u_2 = \sin 3x \qquad dv_2 = e^{5x}dx$$
$$du_2 = 3\cos 3x\, dx \qquad v_2 = \frac{1}{5}\, e^{5x}\ .$$

to get

$$\int e^{5x} \cos 3x \, dx = \frac{1}{5}e^{5x}\cos 3x + \frac{3}{5}\left[\frac{1}{5}e^{5x}\sin 3x - \frac{3}{5}\int e^{5x}\cos 3x \, dx\right]$$
$$= \frac{1}{5}e^{5x}\cos 3x + \frac{3}{5^2}e^{5x}\sin 3x - \frac{3^2}{5^2}\int e^{5x}\cos 3x \, dx \; .$$

Now solving for $\int e^{5x} \cos 3x \, dx$ (use the *bring to the other side* idea) we get

$$\left[1 + \frac{3^2}{5^2}\right]\int e^{5x} \cos 3x \, dx = \frac{1}{5}e^{5x}\cos 3x + \frac{3}{5^2}e^{5x}\sin 3x + K$$

and so

$$\int e^{5x} \cos 3x \, dx = \left[\frac{5^2}{5^2+3^2}\right]\left(\frac{1}{5}e^{5x}\cos 3x + \frac{3}{5^2}e^{5x}\sin 3x + K\right)$$
$$= \frac{5}{34}\, e^{5x}\cos 3x + \frac{3}{34}\, e^{5x}\sin 3x + \left[\frac{K5^2}{5^2+3^2}\right]$$
$$= \frac{e^{5x}}{34}\,(5\cos 3x + 3\sin 3x) + \left[\frac{K5^2}{5^2+3^2}\right]$$

Thus

$$\int e^{5x} \cos 3x \, dx = \frac{e^{5x}}{34}\,(5\cos 3x + 3\sin 3x) + C \; .$$

Doesn't Work Way

If you try two integration by part with letting the exponential function be with the u one time and the dv the other time, then when you use the *bring to the other side* idea, you will get $0 = 0$, which is true but not helpful.

Next, the diligent student (are you?) will check that

$$D_x\left[\frac{e^{5x}}{34}\,(5\cos 3x + 3\sin 3x)\right] = e^{5x}\,\cos 3x$$

So

$$\int_0^{\pi} e^{5x}\cos 3x \, dx = \frac{e^{5x}}{34}\,(5\cos 3x + 3\sin 3x)\,|_0^{\pi}$$
$$= \left[\frac{e^{5\pi}}{34}\,(5\cos 3\pi + 3\sin 3\pi)\,|_0^{\pi}\right] - \left[\frac{e^{0}}{34}\,(5\cos 0 + 3\sin 0)\,|_0^{\pi}\right]$$
$$= \left[\frac{e^{5\pi}}{34}\,(-5+0)\right] - \left[\frac{1}{34}\,(5+0)\right] = \frac{-5}{34}\left(e^{5\pi}+1\right)$$

5. Evaluate the integral

$$\int_{-1}^{0} \frac{1}{\left(x^2+2x+2\right)^2}\,dx \; .$$

Hint. Complete the square: $x^2 + 2x + 2 = (x \pm \;?\;)^2 \pm \;??$.

5soln.

$$\int \frac{1}{[x^2+2x+2]^2}\,dx = \int \frac{1}{[(x+1)^2+1]^2}\,dx = \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \int \cos^2\theta\,d\theta$$

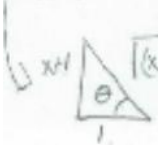
$x+1 = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$(x+1)^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\,\frac{2\cos\theta\sin\theta}{2} + C = \frac{1}{2}\theta + \frac{1}{2}\cos\theta\sin\theta + C$$



$\sqrt{(x+1)^2+2} = \sqrt{x^2+2x+2}$

$$= \frac{1}{2}\arctan(x+1) + \frac{1}{2}\cdot\frac{x+1}{x^2+2x+2} + C$$

So

$$\int_{-1}^{0} \frac{dx}{(x^2+2x+2)^2} = \left(\frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\,\frac{x+1}{x^2+2x+2}\right)\Bigg|_{x=-1}^{x=0}$$

$$= \left[\frac{1}{2}\underbrace{\tan^{-1}(1)}_{=\frac{\pi}{4}} + \frac{1}{2}\,\frac{1}{2}\right] - \left[\frac{1}{2}\underbrace{\tan^{-1}(0)}_{=0} + \frac{1}{2}\,0\right] = \frac{\pi}{8} + \frac{1}{4}$$

6. Let $y = p(x)$ be a polynomial of degree 5.

What is the form of the partial fraction decomposition of

$$\frac{p(x)}{(x^2-1)\,(x^2+1)^2}\ ?$$

Here A, B, C, D, E and F are constants.

6soln. $(x^2-1)\,(x^2+1) = (x-1)\,(x+1)\,(x^2+1)$ where $x-1$ and $x+1$ are linear terms while x^2+1 is an irreducible quadratic term. The partial fraction lecture/handout from class explains why the PDF takes the form $\dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$.

7. Evaluate the integral

$$\int_1^3 \frac{5x^2+3x-2}{x^3+2x^2}\,dx \qquad .$$

7soln. Partial Fraction Decomposition Problem.

As usual,

(1) we first find the indefinite integral,

(2) then check that our indefinite integral is correct by integrating the indefinite integral and making sure we get the integrand,

(3) and then evaluate our definite integral.

- $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So $\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right) dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C$.

- Check $D_x\left[2\ln|x| + x^{-1} + 3\ln|x+2|\right] = \frac{2}{x} + -1x^{-2} + \frac{3}{x+2}$
$= \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2} = \frac{2x(x+2) - (x+2) + 3x^2}{x^2(x+2)} = \frac{5x^2+3x-2}{x^3+2x^2}$ ✓

- $\left[3\ln|x+2| + 2\ln x + \frac{1}{x}\right]\Big|_{x=1}^{x=3} =$
$\left[3\ln 5 + 2\ln 3 + \frac{1}{3}\right] - \left[3\ln 3 + \underbrace{2\ln 1}_{=0} + 1\right] =$
$3\ln 5 - \ln 3 - \frac{2}{3}$.

8. Evaluate the integral

$$\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx\ .$$

8soln. $\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx$ does not exist but also does not diverge to infinity.

- $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$

$\int_{x=0}^{x=1} x^{-3}\,dx = \lim_{a\to 0^+} \frac{x^{-2}}{-2}\Big|_{x=a}^{x=1} = \frac{1}{2}\lim_{a\to0^+}\left[\frac{1}{x^2}\right]_{x=1}^{x=a} =$
$\frac{1}{2}\lim_{x\to 0^+}\left[\frac{1}{a^2} - 1\right] = \infty$. Similarly, $\int_{-1}^{0} x^{-3}\,dx = -\infty$.

- $\int_{-1}^{1} x^{-3}\,dx = \int_{-1}^{0} x^{-3}\,dx + \int_{0}^{1} x^{-3}\,dx = -\infty + \infty$ so DNE.

9. Evaluate the integral

$$\int_{x=-\infty}^{x=\infty} \frac{1}{1+x^2}\,dx\ .$$

9soln. $\int_{x=-\infty}^{x=\infty} \frac{1}{1+x^2}\,dx = \pi$. From our textbook, page 506, Example 2.

HISTORICAL BIOGRAPHY
Lejeune Dirichlet
(1805–1859)

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} \\ &= \lim_{a\to-\infty} \tan^{-1} x \Big]_a^0 \\ &= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\ &= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

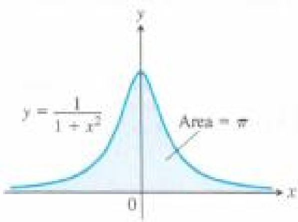


FIGURE 8.15 The area under this curve is finite (Example 2).

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15).

10. Let c be a real number. Evaluate, if it exists, the limit of the sequence

$$\lim_{n\to\infty} \left(1 + \tfrac{c}{n}\right)^n$$.

10soln.

10 $\left(1 + \frac{c}{n}\right)^n$ $\langle$ as $n \to \infty$ $\to 1^\infty$, ind. form, so... $\rangle$ (↑ thinking land)

Let $y = \left(1 + \frac{c}{x}\right)^x$. $\langle \to 1^\infty$, indet. form so... $\rangle$

So $\ln y = x \ln\left(1 + \frac{c}{x}\right) = \dfrac{\ln\left(1 + \frac{c}{x}\right)}{\frac{1}{x}}$.

$$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{c}{x}} \cdot D_x \frac{c}{x}}{D_x \frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{x+c}{x}} (\cdot c) = \lim_{x\to\infty} \frac{cx}{x+c} = c$$

$$\Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^c$$

11. Evaluate

$$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$$

Hint: Telescoping Series, use PFD.

11soln.

$$\frac{4}{(4n-3)(4n+1)} \overset{\text{PFD}}{=} \frac{A}{4n-3} + \frac{B}{4n+1} = \frac{A(4n+1)+B(4n-3)}{(4n-3)(4n+1)}$$

$$4 = A(4n+1) + B(4n-3)$$

$n^1 :\quad 0 = 4A + 4B \implies A = -B$

$n^0 :\quad 4 = A - 3B$

$\implies 4 = -B - 3B = -4B \implies B = -1 \implies A = -B = 1$

$$S_N = \sum_{n=1}^{N} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{N} \left[\frac{1}{4n-3} + \frac{-1}{4n+1} \right]$$

$= \frac{1}{1} + \frac{-1}{5}$ ← $n=1$

$+ \frac{1}{5} + \frac{-1}{9}$ ← $n=2$

$+ \frac{1}{9} + \frac{-1}{13}$ ← $n=3$

$+ \frac{1}{13} + \frac{-1}{17}$ ← $n=4$

⋮ yes... we see the pattern & the cancellations

$\frac{1}{4N-3} + \frac{-1}{4N+1}$ ← $n=N$

$$= 1 + \frac{-1}{4N+1} \xrightarrow{N\to\infty} \boxed{1}.$$

12. Consider the formal series $\sum_{n=1}^{\infty} a_n$ where

$$a_n \;=\; (-1)^n \,\frac{(n+1)!}{(2n)!}$$

and let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \;.$$

12soln.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{[(n+1)+1]!}{[2(n+1)]!} \cdot \frac{(2n)!}{(n+1)!}$$

< collect up like terms >

$$= \frac{(n+2)!}{(n+1)!} \; \frac{(2n)!}{(2n+2)!}$$

< simplify ... cancel >

$$= \frac{(n+1)!\,(n+2)}{(n+1)!} \; \frac{(2n)!}{(2n)!\,(2n+1)\,(2n+2)}$$

$$= \frac{n+2}{(2n+1)(2n+2)} = \frac{n+2}{4n^2 + ?\,n^1 + ?\,n^0} \xrightarrow{n\to\infty} 0 < 1.$$

$$0 \equiv \rho$$

abs. conv by ratio test.

13. Consider the formal series $\sum_{n=1}^{\infty} a_n$ where

$$a_n \;=\; \frac{\sqrt{n+2}}{2n^2+n+1} \;.$$

13soln. First note that each $a_n > 0$ so $\sum a_n$ is a positive series. When n is big,

$$a_n \;=\; \frac{\sqrt{n+2}}{2n^2+n+1} \overset{n\text{big}}{\sim} \frac{\sqrt{n}}{2n^2} = \frac{1}{2}\frac{n^{1/2}}{n^2} = \frac{1}{2}\frac{1}{n^{3/2}}$$

so we will use a comparison test with

$$b_n = \frac{1}{n^{3/2}}$$

noting that $\sum b_n = \sum \frac{1}{n^{3/2}}$ is a p-series, with $p = \frac{2}{3} > 1$ and so $\sum b_n$ converges. So

$$\frac{a_n}{b_n} = \frac{\frac{\sqrt{n+2}}{2n^2+n+1}}{\frac{1}{n^{3/2}}} = \frac{n^{3/2}\,(n+2)^{1/2}}{2n^2+n+1} \underset{\text{is } n^2}{\overset{\text{dominating force}}{=}} \frac{\frac{n^{3/2}}{n^{3/2}}\frac{(n+2)^{1/2}}{n^{1/2}}}{\frac{2n^2+n+1}{n^2}} = \frac{\left(\frac{n}{n}\right)^{3/2}\left(\frac{n+2}{n}\right)^{1/2}}{2+\frac{1}{n}+\frac{1}{n^2}}$$

$$= \frac{(1)^{3/2}\left(1+\frac{2}{n}\right)^{1/2}}{2+\frac{1}{n}+\frac{1}{n^2}} \xrightarrow{n\to\infty} \frac{1}{2} := L\,.$$

Since $a_n, b_n > 0$ and $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{2} := L$ and $0 < L < \infty$, by the LCT, $\sum a_n$ and $\sum b_n$ *do the same thing*. We know that $\sum b_n = \sum \frac{1}{n^{3/2}}$ is a p-series, with $p = \frac{3}{2} > 1$, and so $\sum b_n$ converges. So $\sum a_n$ also converges.

14. What is the LARGEST interval (so you have to check your endpoints) for which the formal power series

$$\sum_{n=1}^{\infty} \frac{(5x+15)^n}{4^n}$$

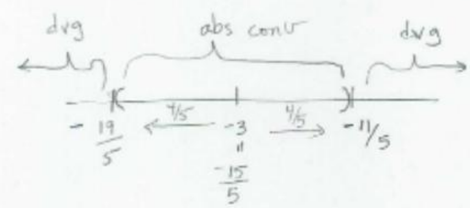
is absolutely convergent?

14soln.

$$\sum \frac{(5x+15)^n}{4^n} = \sum \frac{[5(x+3)]^n}{4^n} = \sum \left(\frac{5}{4}\right)^n (x - -3)^n \;.\; \Rightarrow \text{center} =$$

$$\left[\left|\frac{(5x+15)^n}{4^n}\right|\right]^{1/n} = \left|\frac{5x+15}{4}\right| = \frac{5}{4}|x+3| \xrightarrow{n\to\infty} \frac{5}{4}|x+3|$$

$$\frac{5}{4}|x+3| < 1 \iff |x+3| < \frac{4}{5} \Rightarrow \text{rad. of conv. is } \frac{4}{5}$$



Check endpts

$$x = -11/5 :\; \sum \frac{(5x+15)^n}{4^n} = \sum \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1 \quad \underline{\text{dvg}} \;(\text{to } \infty)$$

$$x = -19/5 :\; \sum \frac{(5x+15)^n}{4^n} = \sum \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n \quad \text{dvg (osc.)}$$

15. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) = \frac{2}{3-x}$$

about the center $x_0 = 0$ and state when this Taylor series is valid.

15soln.

$$f(x) = \frac{2}{3-x} = \frac{2}{3}\left[\frac{1}{1-\left(\frac{x}{3}\right)}\right] \overset{(GS)}{=} \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2}{3^{n+1}}x^n .$$

The Geometric Series expansion (GS) is valid $\iff \left|\frac{x}{3}\right| < 1 \iff |x| < 3$

16. Using a known (commonly used) Taylor series, find the Taylor series for

$$f(x) = \frac{1}{(1-x)^4}$$

about the center $x_0 = 0$ which is valid for $|x| < 1$.

Hint. Start with the Geometric Series (Prof. Girardi sometimes called him the work moose) and differentiate (as many times as needed). Be careful and don't forget the chain rule:

$$D_x(1-x)^{-1} = (-1)(1-x)^{-2}\,D_x(1-x) = (-1)(1-x)^{-2}\,(-1) = (1-x)^{-2} .$$

16soln.

Start with Geometric Series and take Derivatives as many times as need.

Geometric Series is valid when $|x| < 1$ so resulting power series expansions will also be valid when $|x| < 1$.

Geometric Series $\Rightarrow$ $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \overset{D_x}{\Longrightarrow} (1-x)^{-2} = \sum_{k=1}^{\infty} k\,x^{k-1}$

$\overset{D_x}{\Longrightarrow} 2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1)\,x^{k-2} \overset{D_x}{\Longrightarrow} 2\cdot 3(1-x)^{-4} = \sum_{k=3}^{\infty} k(k-1)(k-2)\,x^{k-3}$

So

$$(1-x)^{-4} = \sum_{k=3}^{\infty}\frac{k(k-1)(k-2)}{6}x^{k-3} = \sum_{n=0}^{\infty}\frac{(n+3)(n+2)(n+1)}{6}x^n$$

let $k-3=n$ $\Rightarrow k = n+3$

17. Consider the function

$$f(x) = e^{-x}$$

over the interval $(7, 9)$. The 5^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_5(x) = \sum_{n=0}^{5} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} .$$

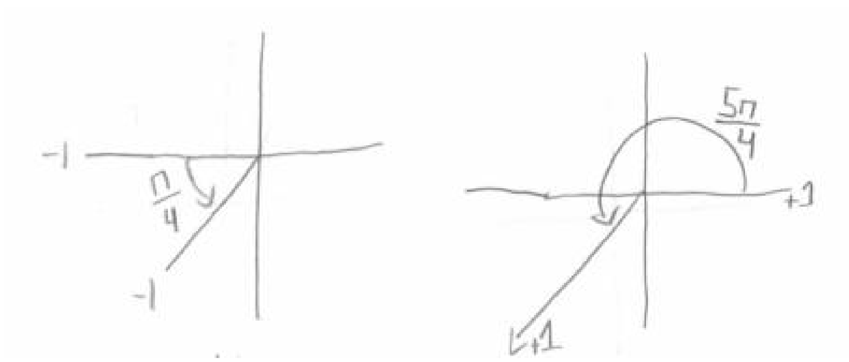
The 5^{th} order Remainder term $R_5(x)$ is defined by $R_5(x) = f(x) - P_5(x)$ and so $e^{-x} \approx P_5(x)$ where the approximation is within an error of $|R_5(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_5(x)|$ that is valid for each $x \in (7, 9)$.

17soln. In the below, $x \in (7, 9)$ and c is between x and 0 and so $c \in (0, 9)$.

$$|R_5(x)| \leq \left| \frac{f^{(6)}(c)}{6!} (x - x_0)^6 \right| = \frac{e^{-c}}{6!} |x|^6 \leq \frac{e^{-0}}{6!} 9^6 = \frac{9^6}{6!} .$$

18. The point with a polar coordinate representation $\left({}^{-}1, \frac{\pi}{4} \right)$ also has a polar coordinate representation $\left({}^{+}1, \theta \right)$ where θ is

18soln.



A point with polar coordinates (r, θ) has Cartesian coordinates $(r \cos\theta, r \sin\theta)$.

So we want to find θ so that

$$(+1)\cos\theta \overset{\text{want}}{=} (-1)\cos\frac{\pi}{4} \overset{\text{know}}{=} \frac{-\sqrt{2}}{2}$$

and

$$(+1)\sin\theta \overset{\text{want}}{=} (-1)\sin\frac{\pi}{4} \overset{\text{know}}{=} \frac{-\sqrt{2}}{2}$$

So we can take $\theta = \frac{5\pi}{4}$.

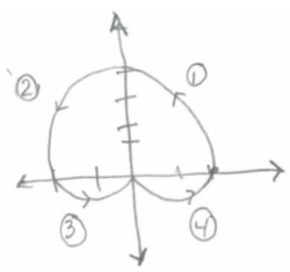
19. Express the arc length of the *heart* traced out by the curve given by (in polar coordinates)

$r = 2 + 2\sin\theta$

as an integral.

19soln.

helpful table for graphing				
period of $\sin(\mathbf{1}\theta) = \frac{2\pi}{\mathbf{1}} = 2\pi$ and so $\frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2} = \Delta\theta$				
	θ	$\sin\theta$	$2\sin\theta$	$r = 2 + 2\sin\theta$
①	$0 \to \frac{\pi}{2}$	$0 \to 1$	$0 \to 2$	$2 \to 4$
②	$\frac{\pi}{2} \to \pi$	$1 \to 0$	$2 \to 0$	$4 \to 2$
③	$\pi \to \frac{3\pi}{2}$	$0 \to -1$	$0 \to -2$	$2 \to 0$
④	$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$-2 \to 0$	$0 \to 2$



Note the heart is traced out <u>once</u> as θ goes from 0 to 2π.

So let $\alpha = 0$ and $\beta = 2\pi$

So the arc length, expressed as an integral, is

$$AL = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \boxed{\int_0^{2\pi} \sqrt{(2 + 2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta}$$

20. Prof. Girardi likes

ⓐ the number 17

ⓑ moose

ⓒ colored chalk

ⓓ mathematics

ⓔ All of the above.

Good Luck in your math fun to come!