HAND IN PART

MARK BOX		
PROBLEM	POINTS	
0	15	
1-12	60	
13	10	
14	15	
%	100	

NAME: Solutions

PIN: 17

INSTRUCTIONS

- This exam comes in two parts.
  1. HAND IN PART. Hand in only this part.
  2. STATEMENT OF MULTIPLE CHOICE PROBLEMS. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- **On Problem 0**, fill in the blanks. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For multiple choice problems 1–12**, circle your answer(s) on the provided chart. No need to show work. The STATEMENT OF MULTIPLE CHOICE PROBLEMS will not be collected.
- **For problems > 12**, to receive credit you **MUST**:
  1. **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
  2. if a line/box is provided, then:
     — show you work BELOW the line/box
     — put your answer on/in the line/box
  3. if no such line/box is provided, then box your answer.
- The MARK BOX above indicates the problems along with their points.
  Check that your copy of the exam has all of the problems.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: books, electronic devices, any device with which you can connect to the internet, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. At a student's request, I will project my watch upon the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Thomas, 13^{th} ed., ET): §8.1–8.5 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________

0. Fill in the blanks (each worth 1 point). (Recall that the *integrand* is the function you are integrating.)

0.1 $\int \frac{du}{u}$ = $\ln|u|$ +C

0.2 $\int \cos u\, du$ = $\sin u$ +C

0.3 $\int \sin u\, du$ = $-\cos u$ +C

0.4 $\int \sec^2 u\, du$ = $\tan u$ +C

0.5 $\int \tan u\, du$ = $\ln|\sec u| \overset{\text{or}}{=} -\ln|\cos u|$ +C

0.6 $\int \cot u\, du$ = $-\ln|\csc u| \overset{\text{or}}{=} \ln|\sin u|$ +C

0.7 If a is a contant and $a > 0$ then $\int \frac{1}{\sqrt{a^2-u^2}}\, du$ = $\sin^{-1}\left(\frac{u}{a}\right)$ +C

0.8 If a is a contant and $a > 0$ then $\int \frac{1}{a^2+u^2}\, du$ = $\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$ +C

0.9 Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do long division

0.10 Integration by parts formula: $\int u\, dv$ = $uv - \int v du$

0.11 Trig. Substitution:
if the integrand involves $a^2 + u^2$, then one makes the substitution u = $a\tan\theta$

0.12 Trig. Substitution:
if the integrand involves $u^2 - a^2$, then one makes the substitution u = $a\sec\theta$

0.13 Trig. Substitution:
if the integrand involves $a^2 - u^2$, then one makes the substitution u = $a\,\sin\theta$

0.14 Trig. Formula. (your answer should involve trig functions of θ, and not of 2θ): $\sin(2\theta) = 2\sin\theta\cos\theta$

0.15 Trig Formula. Since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween tangent (i.e., tan) and secant (i.e., sec) is $1 + \tan^2\theta = \sec^2\theta$

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS

- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 5 points.
  - For a problem with precisely two answers marked, one of which is correct, 3 points.
  - For a problem with nothing marked (i.e., left blank) 1 point.
  - All other cases, 0 points.
- Fill in the "number of solutions circled" column. (Worth a total of 1 point of extra credit.)

Your Solutions							Do Not Write Below			
PROBLEM						number of solutions circled	1	2	B	x
1	1a	1b	1c	1d	1e					
2	2a	2b	2c	2d	2e					
3	3a	3b	3c	3d	3e					
4	4a	4b	4c	4d	4e					
5	5a	5b	5c	5d	5e					
6	6a	6b	6c	6d	6e					
7	7a	7b	7c	7d	7e					
8	8a	8b	8c	8d	8e					
9	9a	9b	9c	9d	9e					
10	10a	10b	10c	10d	10e					
11	11a	11b	11c	11d	11e					
12	12a	12b	12c	12d	12e					
							Extra Credit:			

13. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + \text{C}$

13soln.

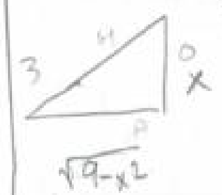
let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$\frac{x}{3} = \sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

SOHCAH



$$= \int \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin\theta)^2}}\,3\cos\theta\, d\theta = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\cancel{3\cos\theta}}\,\cancel{3\cos\theta}\, d\theta$$

$$= \int 9\sin^2\theta\, d\theta$$

$$= 9\int \sin^2\theta\, d\theta$$

$$= 9\int \tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\int (\theta - \cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}2\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C$$

Dear Math 142 students,

Do you recognize this problem?

Jonathan, our SI Leader, had it on your SI Practice Exam 1.

SI is such a wonderful opportunity for LLSC students!

Prof. Girardi

14. $$\boxed{\int \frac{x+4}{x^2+2x+5}\,dx \;=\; \frac{1}{2}\,\ln\left|x^2+2x+5\right| \;+\; \frac{3}{2}\,\arctan\left(\frac{x+1}{2}\right) \qquad + \text{C}}$$

Way # 1

Shall we try PDF? To check if the denominator $y = x^2 + 2x + 5$ is an irreducible quadratic (IQ) or a product of two linear terms, let's check its discriminant:

$$b^2 - 4ac \;=\; (2)^2 - (4)(1)(5) \;<\; 0\,.$$

So $y = x^2 + 2x + 5$ is an irreducible quadrant (IQ). Thus the PDF of the integrand is of the form

$$\frac{x+4}{x^2+2x+5} \;=\; \frac{Ax+B}{x^2+2x+5}\,. \tag{1}$$

It's easy to solve for the constants A and B in (1): $A = 1$ and $B = 4$. I.e., the integrand is already in its PFD!

Aiming for a $\int \frac{du}{u}$, let u be the denominator of the integrand:

$$\begin{aligned} u &= x^2 + 2x + 5 \\ du &= (2x+2)\,dx \\ du &= 2\,(x+1)\,dx \end{aligned}$$

By completing the square we have

$$x^2 + 2x + 5 \;=\; (x+1)^2 + 4 \;=\; (x+1)^2 + 2^2 \tag{2}$$

and we know that

$$\int \frac{du}{a^2+u^2} \;=\; \frac{1}{a}\,\arctan\left(\frac{u}{a}\right)\,.$$

Now to put it all together.

$$\begin{aligned} \int \frac{(x+4)\,dx}{x^2+2x+5} &= \int \frac{(x+1)\,dx}{x^2+2x+5} && + \int \frac{(3)\,dx}{x^2+2x+5} \\ &= \frac{1}{2}\int \frac{\boxed{2\,(x+1)\,dx}}{x^2+2x+5} && + \;3\int \frac{dx}{(x+1)^2+2^2} \\ & && \quad\downarrow \text{let } t = x+1 \\ &= \frac{1}{2}\int \frac{du}{u} && + \;3\int \frac{dt}{t^2+2^2} \\ &= \frac{1}{2}\,\ln|u| && + \;3\left[\frac{1}{2}\,\arctan\left(\frac{t}{2}\right)\right] + C \\ &= \frac{1}{2}\,\ln\left|x^2+2x+5\right| && + \;3\left[\frac{1}{2}\,\arctan\left(\frac{x+1}{2}\right)\right] + C\,. \end{aligned}$$

Way # 2

From (2) we see that

$$\int \frac{x+4}{x^2+2x+5}\,dx \;=\; \frac{(x+4)}{(x+1)^2+2^2}\,dx$$

and so we let $x+1 = 2\tan\theta$.

$$\begin{aligned} x+1 &= 2\tan\theta \\ dx &= 2\sec^2\theta\, d\theta \end{aligned}$$

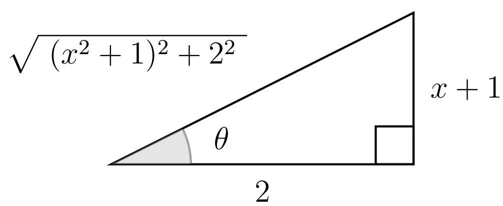
$$(x+1)^2 + 2^2 = (2\tan\theta)^2 + 4 = 4\tan^2\theta + 4 = 4\left(\tan^2\theta + 1\right) = 4\sec^2\theta \ .$$

Thus

$$\begin{aligned} \int \frac{x+4}{x^2+2x+5}\, dx &= \int \frac{(x+1)+3}{(x+1)^2+2^2}\, dx \\ &= \int \frac{(2\tan\theta)+3}{4\sec^2\theta} \left(2\sec^2\theta\right) d\theta \\ &= \frac{1}{2}\int (2\tan\theta + 3)\ d\theta \\ &= \int \tan\theta\, d\theta \ + \ \frac{3}{2}\int d\theta \\ &= \ln|\sec\theta| \ + \ \frac{3\theta}{2} + C \ . \end{aligned}$$

Now we make a reference triangle using that $2\tan\theta = x+1$.

$$\tan\theta = \frac{x+1}{2} \qquad \Longrightarrow$$



Thus Thus

$$\int \frac{x+4}{x^2+2x+5}\, dx = \ln\left|\frac{\sqrt{(x^2+1)^2+4}}{2}\right| \ + \ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) \ + C \ .$$

Compare the two ways.

If you compare the two ways, you'll see they are very similar. Do you see why? Note that by the properties of the log functions, the two answers are the same since

$$\begin{aligned} &\ln\left|\frac{\sqrt{(x^2+1)^2+4}}{2}\right| \ + \ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) \ + C_2 \\ &= \left[\ln\left|\sqrt{x^2+2x+5}\right| - \ln 2\right] \ + \ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) \ + C_2 \\ &= \ln\left|\left(x^2+2x+5\right)^{1/2}\right| \ + \ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) \ + [C_2 - \ln 2] \\ &= \ln\left|x^2+2x+5\right|^{1/2} \ + \ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) \ + [C_2 - \ln 2] \\ &= \frac{1}{2}\ln\left|x^2+2x+5\right| \ + \ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) \ + [C_2 - \ln 2] \\ &= \frac{1}{2}\ln\left|x^2+2x+5\right| \ + \ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) \ + C_1 \end{aligned}$$

where $C_1 = C_2 - \ln 2$.

STATEMENT OF MULTIPLE CHOICE PROBLEMS
These sheets of paper are not collected.

- Hint. For a definite integral problems $\int_a^b f(x)\,dx$.
  (1) First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$.
  (2) Next check if you did the indefininte integral correctly by using the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$).
  (3) Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint. If $a, b > 0$ and $r \in \mathbb{R}$, then $\ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$.
- Hint. Throughout this exam, unless otherwise stated, follow the common calculus practice of measuring angles in radians (not degrees).

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1. Evaluate the integral

$$\int_{x=0}^{x=1} \frac{1}{x^2+1}\,dx\ .$$

a. $\ln\sqrt{2}$
b. $\ln 2$
c. $\frac{\pi}{4}$
d. $\frac{\pi}{2}$
e. None of the others.

1soln. $\int_{x=0}^{x=1} \frac{1}{x^2+1}\,dx \ = \arctan x\big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

2. The integral

$$\int \frac{x}{x^2+1}\,dx$$

can be evaluated the following way.
a. Trig. Substitution using the $x = \sin\theta$.
b. A simple *u*-*du* substitution with $u = x^2 + 1$.
c. The integrand is not in its Partial Fraction Decomposition (PDF) so find the integrand's PDF, for which long division is not necessary.
d. The integrand is not in its Partial Fraction Decomposition (PDF) so find the integrand's PDF, for which long division is necessary.
e. None of the others.

2soln. If $u = x^2+1$, then $\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\int \frac{2x\ dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$.
One can integrate the integral using simple u-du substitution of $u = x^2+1$.
Note that the integrand is already in its Partial Fraction Decomposition.

3. Evaluate the integral

$$\int_0^4 \frac{x}{x+9}\, dx \ .$$

a. $4 - 9\ln(13) + 9\ln(9)$
b. $13 - 9\ln(4) + \ln(3)$
c. $\frac{1}{9\ln(13)} - \ln(3)$
d. $4 - 13\ln(9) + 3\ln(18)$
e. None of the others.

3soln.

$$\int \frac{x}{x+9}\,dx = \int 1dx - 9\int \frac{dx}{x+9} = x - 9\ln|x+9| + C$$

$$\boxed{\frac{x}{x+9} = \frac{x+9}{x+9} - \frac{9}{x+9} = 1 - \frac{9}{x+9}}$$

Long Division (Fake)

Check $D_x\left[x - 9\ln|x+9|\right] = 1 - \frac{9}{x+9}$
$= \frac{x+9}{x+9} - \frac{9}{x+9} = \frac{x}{x+9}$. ✓

So $\int_0^4 \frac{x}{x+9}\,dx = \left[x - 9\ln|x+9|\right]\Big|_{x=0}^{x=4}$
$= \left[4 - 9\ln|13|\right] - \left[0 - 9\ln|9|\right]$
$= 4 - 9\ln(13) + 9\ln(9)$

4. Evaluate the integral

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x\ dx \ .$$

a. $\frac{-1}{3}$
b. $\frac{-2}{3}$
c. $\frac{1}{3}$
d . $\frac{2}{3}$
e. None of the others.

4soln.

$$\int \cos^3 x\, dx = \int (1-\sin^2 x)\cos x\, dx \overset{u=\sin x}{=} \int (1-u^2)\, du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x\, dx = \left(\sin x - \frac{1}{3}\sin^3 x\right)\Big|_{x=0}^{x=\frac{\pi}{2}} = \left(1-\frac{1}{3}\right) - (0-0) = \frac{2}{3}\ .$$

5. Evaluate the integral

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^2 x\ dx\ .$$

a. $\frac{\pi}{4}$

b. π

c. $\frac{3\pi}{4}$

d. 2π

5soln. Since

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x \cdot 2dx$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^2 x\ dx \ = \left(\frac{x}{2} + \frac{1}{4}\sin 2x\right)\Big|_{x=0}^{x=\frac{\pi}{2}} = \left(\frac{\pi}{4}+0\right) + (0+0) = \frac{\pi}{4}$$

6. Evaluate the integral

$$\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x\ \sin^4 x\ dx\ .$$

a. $\dfrac{4}{45}$

b. $\dfrac{14}{45}$

c. $\dfrac{2}{35}$

d. $\dfrac{12}{35}$

e. None of the others.

6soln. $\displaystyle\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x \, \sin^4 x \; dx \;=\; (\tfrac{1}{5} - \tfrac{1}{7}) - (0-0) = \tfrac{2}{35}$.

Example 4 Evaluate $\int \cos^3 x \sin^4 x \, dx$.

Solution We proceed as follows:

$$\begin{aligned}\int \cos^3 x \sin^4 x \, dx &= \int \cos^2 x \sin^4 x \cos x \, dx \\ &= \int (1 - \sin^2 x) \sin^4 x \cos x \, dx.\end{aligned}$$

If we let $u = \sin x$, then $du = \cos x \, dx$ and the integral may be written

$$\begin{aligned}\int \cos^3 x \sin^4 x \, dx &= \int (1 - u^2) u^4 \, du \\ &= \int (u^4 - u^6) \, du \\ &= \tfrac{1}{5} u^5 - \tfrac{1}{7} u^7 + C \\ &= \tfrac{1}{5} \sin^5 x - \tfrac{1}{7} \sin^7 x + C.\end{aligned}$$

7. Evaluate the integral

$$\int_{x=1}^{x=e} \ln x \; dx \; .$$

a. 0

b. 1

c. e

d. $2e$

e. None of the others.

7soln. Since

EXAMPLE 2 Evaluate $\int \ln x\, dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x}\, dx \qquad v = x$$

Integrating by parts, we get

$$\begin{aligned}\int \ln x\, dx &= x \ln x - \int x \frac{dx}{x} \\ &= x \ln x - \int dx \\ &= x \ln x - x + C\end{aligned}$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f.

$$\int_{x=1}^{x=e} \ln x \; dx \;= (x \ln x - x)\,\Big|_{x=1}^{x=e} = (e \ln e - e) - (1 \ln 1 - 1) = (e - e) - (0 - 1) = 1$$

8. Evaluate the integral

$$\int\limits_{x=0}^{x=\pi} e^{3x} \cos 2x \; dx \; .$$

a. 0
b. $\frac{1}{5}\left(3e^{3\pi}\right)$
c. $\frac{1}{5}\left(3e^{3\pi} - 3\right)$
d. $\frac{1}{13}\left(3e^{3\pi} - 3\right)$
e. None of the others.

8soln. This problem is based upon a homework problem, which Michael turned into a reciation quiz. His quizzes are good prepartions for exams.
Two integration by parts and the *bring to the other side* idea. For both the integration by parts, either let both u's involve the expontential function or let both dv's involve the expontential function; we'll do the latter way. So let

$$\begin{aligned} u_1 &= \cos 2x & dv_1 &= e^{3x} dx \\ du_1 &= -2 \sin 2x \, dx & v_1 &= \frac{1}{3} e^{3x} \; . \end{aligned}$$

So, by integration by parts

$$\int e^{3x} \cos 2x \; dx \;=\; \frac{1}{3} e^{3x} \cos 2x - \frac{-2}{3} \int e^{3x} \sin 2x \, dx \; .$$

Now let

$$u_2 = \sin 2x \qquad dv_2 = e^{3x} dx$$
$$du_2 = 2\cos 2x\, dx \qquad v_2 = \frac{1}{3}\, e^{3x} \;.$$

to get

$$\begin{aligned}\int e^{3x}\cos 2x\; dx &= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\left[\frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x\, dx\right] \\ &= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3^2}e^{3x}\sin 2x - \frac{2^2}{3^2}\int e^{3x}\cos 2x\, dx \;.\end{aligned}$$

Now to solve for $\int e^{3x}\cos 2x\, dx$ (use the *bring to the other side* idea) we have

$$\left[1 + \frac{2^2}{3^2}\right]\int e^{3x}\cos 2x\; dx = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3^2}e^{3x}\sin 2x + C$$

and so

$$\begin{aligned}\int e^{3x}\cos 2x\; dx &= \left[\frac{3^2}{3^2+2^2}\right]\left(\frac{1}{3}e^{3x}\cos 2x + \frac{2}{3^2}e^{3x}\sin 2x + C\right) \\ &= \frac{3}{13}\,e^{3x}\cos 2x + \frac{2}{13}\,e^{3x}\sin 2x + \left[\frac{C3^2}{3^2+2^2}\right] \\ &= \frac{e^{3x}}{13}\,(3\cos 2x + 2\sin 2x) + \left[\frac{C3^2}{3^2+2^2}\right]\end{aligned}$$

Thus

$$\begin{aligned}\int_{x=0}^{x=\pi} e^{3x}\cos 2x\; dx &= \frac{e^{3x}}{13}\,(3\cos 2x + 2\sin 2x)\,\Big|_{x=0}^{x=\pi} = \left[\frac{e^{3\pi}}{13}\,(3+0)\right] - \left[\frac{e^{0}}{13}\,(3+0)\right] \\ &= \frac{3}{13}\left(e^{3\pi}-1\right).\end{aligned}$$

Clearly

$$\frac{3}{13}\left(e^{3\pi}-1\right) = \frac{1}{13}\left(3e^{3\pi}-3\right) \;.$$

9. Evaluate the integral

$$\int_{x=0}^{x=1} \frac{1}{\sqrt{4+x^2}}\; dx \;.$$

Do not overlook the square root sign in the denominator.

a. $\frac{1}{2}\arctan\frac{1}{2}$

b. $\arctan\frac{1}{2}$

c. $\ln\left|\frac{\sqrt{5}}{2}+\frac{1}{2}\right|$

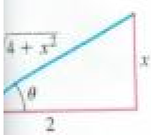
d. $\ln\left|\frac{\sqrt{5}}{2}+\frac{1}{2}\right| - \ln|2|$

e. None of the others.

9soln. For the indefinte integral, let

We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$
$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$



URE 8.4 Reference triangle for = 2 tan θ (Example 1):

$\tan\theta = \frac{x}{2}$

$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$

Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$
$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C. \qquad \text{From Fig. 8.4}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

So

$$\int_{x=0}^{x=1} \frac{1}{\sqrt{4+x^2}}\, dx = \ln\left|\frac{\sqrt{4+1}}{2} + \frac{1}{2}\right| - \ln\left|\frac{\sqrt{4+0}}{2} + \frac{0}{2}\right| = \ln\left|\frac{\sqrt{5}}{2} + \frac{1}{2}\right| - \ln\left|1 + \frac{0}{2}\right| = \ln\left|\frac{\sqrt{5}}{2} + \frac{1}{2}\right|.$$

10. Evaluate the integral

$$\int_{x=0}^{x=\frac{\sqrt{3}}{2}} \frac{4x^2}{(1-x^2)^{3/2}}\, dx$$

AND specify the initial substitution.

a. $\ln\left|(4\sqrt{3} - \frac{4\pi}{3})\right|$ using the initial substitute $x = \sin\theta$.

b. $(4\sqrt{3} - \frac{4\pi}{3})$ using the initial substitute $x = \sin\theta$

c. $\ln\left|(4\sqrt{3} - \frac{4\pi}{3})\right|$ using the initial substitute $x = \sec\theta$.

d. $(4\sqrt{3} - \frac{4\pi}{3})$ using the initial substitute $x = \sec\theta$

e. None of the others.

10soln. Trig Subtitute with initial substitution $x = \sin\theta$. Let

$$x = \sin\theta,\ 0 \le \theta \le \tfrac{\pi}{3},\ dx = \cos\theta\, d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta;$$

$$\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{\left(1-x^2\right)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3}\left(\sec^2\theta - 1\right) d\theta = 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \tfrac{4\pi}{3}$$

11. Let $y = p(x)$ be a polynomial of degree 5.
What is the form of the partial fraction decomposition of

$$\frac{p(x)}{(x^2-1)\,(x^2+1)^2}\ ?$$

Here A, B, C, D, E and F are constants.

a. $\dfrac{A}{x^2-1} + \dfrac{B}{(x^2+1)^2}$

b. $\dfrac{Ax+B}{x^2-1} + \dfrac{Cx+D}{(x^2+1)^2}$

c. $\dfrac{Ax+B}{x^2-1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

d. $\dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

e. None of the others.

11soln. $(x^2-1)\,(x^2+1)^2 = (x-1)\,(x+1)\,(x^2+1)^2$ where $x-1$ and $x+1$ are linear terms while x^2+1 is an irreducible quadratic term. Now see the partial fraction handout from class.

12. Evaluate the integral

$$\int_{x=1}^{x=3} \frac{5x^2+3x-2}{x^3+2x^2}\,dx\ .$$

a. $3\ln 5 - \ln 3 - \frac{2}{3}$

b. $3\ln 5 - \ln 3 - \frac{8}{3}$

c. $\ln 5 - \frac{2}{3}$

d. $\frac{2}{3} - \ln 5$

e. None of the others.

12soln.

- $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to
get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x = -2$ to get $C = 3$, and take
$x = 0$ to get $B = -1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

- Check $D_x\left[2\ln|x| + x^{-1} + 3\ln|x+2|\right] = \dfrac{2}{x} + -1x^{-2} + \dfrac{3}{x+2}$
$= \dfrac{2}{x} - \dfrac{1}{x^2} + \dfrac{3}{x+2} = \dfrac{2x(x+2) - (x+2) + 3x^2}{x^2(x+2)} = \dfrac{5x^2+3x-2}{x^3+2x^2}$ ✓

- $\left[3\ln|x+2| + 2\ln x + \frac{1}{x}\right]\Big|_{x=1}^{x=3} =$

$\left[3\ln 5 + 2\ln 3 + \frac{1}{3}\right] - \left[3\ln 3 + \underbrace{2\ln 1}_{=0} + 1\right] =$

$3\ln 5 - \ln 3 - \frac{2}{3}$.