MARK BOX		
PROBLEM	POINTS	
0	10	
1	10	
2-6	50	
7	15	
8a/8b	15	
%	100	

NAME: KEY-e-poo

PIN: 17

INSTRUCTIONS

- **On Problem 0**, fill in the blanks and boxes. As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
- **For problems 1**, fill in the chart.
- **For problems 2 and 6**, circle your answer(s) on the provided chart. You do **NOT** have to show your work.
- **For problems 7 and 8**, to receive credit you **must** show ALL your work and :
  (1) **work in a logical fashion, show all your work, indicate your reasoning**;
  no credit will be given for an answer that *just appears*;
  such explanations help with partial credit
  (2) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (3) if no such line/box is provided, then box your answer.
- Upon request, you will be given as much (blank) scratch paper as you need.
- Check that your copy of the exam has all of the problems.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: electronic devices, books, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. Please, if I forget, remind me to pull up a clock on the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Stewart, 6^{th} ed., ET): §11.8–11.11 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ____________________

0A. Power Series Consider the (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n\,(x-x_0)^n\ , \tag{1}$$

with radius of convergence $R \in (0,\infty]$. (Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.) Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

- Let $0 < R < \infty$. The largest set of x's for which we know that the power series in (1) is:
  (1) absolutely convergent is $(x_0 - R, x_0 + R)$, also ok: $\{x \in \mathbb{R}\colon\ |x - x_0| < R\}$
  (2) divergent is $(-\infty, x_0 - R) \cup (x_0 + R, \infty)$, also ok: $\{x \in \mathbb{R}\colon\ |x - x_0| > R\}$.

  What can you say about the convergence of the power series in (1) when $x = x_0 + R$ or $x = x_0 - R$?
  the series can be doing anything, i.e., there are examples showing that it can be absolutely convergent, conditionally convergent or divergent .
- Now let $R > 0$ and fill-in the 5 boxes. Consider the function $y = h(x)$ defined by the power series in (1).
  (1) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

  $$h'(x) \;=\; \sum_{n=1}^{\infty} \boxed{n\,a_n\,(x-x_0)^{n-1}}\ . \tag{2}$$

  What can you say about the radius of convergence of the power series in (2)? $\boxed{\text{It's the same } R}$.

  (2) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

  $$\int_{x=\alpha}^{x=\beta} h(x)\,dx \;=\; \sum_{n=0}^{\infty} \boxed{\frac{a_n}{n+1}\,(x-x_0)^{n+1}}\,\Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta}\ .$$

  What can you say about the radius of convergence of the power series in (2)? $\boxed{\text{It's the same } R}$.

0B. Taylor/Maclaurin Polynomials and Series. Fill-in the boxes.
Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

a. In open form (i.e., with "..." notation and without a $\sum$-sign)

$$P_N(x) \;=\; \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N}$$

b. In closed form (i.e., with a $\sum$-sign and without "..." notation)

$$P_\infty(x) \;=\; \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n}$$

c. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) \;=\; \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)}} \quad \text{for some } c \text{ located } \boxed{\text{between } x \text{ and } x_0}\ .$$

d. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.

1. Commonly Used Taylor Series Fill in the LAST COLUMN of the below table. Fill in EITHER (the choice is yours) the open OR closed form COLUMN.

$y = f(x)$	Maclaurin series of $y = f(x)$ in open (with "...") form	Maclaurin series of $y = f(x)$ in closed (with $\sum$-sign) form	Largest set of x's for which the Maclaurin series converges to the function.
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + x^4 + \ldots$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$\mathbb{R}$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!}]$	$\mathbb{R}$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\mathbb{R}$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$	$\sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{x^n}{n}$	$(-1, 1]$

2–6. Instructions for the multiple choice problems 2–6.

- The statement of the multiple choice problems will be handed out separately.
- Indicate (by circling) directly in the table below your solution to these problems.
- You may choice up to **2** answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 10 points.
  - For a problem with precisely two answers marked, one of which is correct, 5 points.
  - All other cases, 0 points.
- Fill in the "number of solutions circled" column.
- You do **NOT** show your work for these problems.

Your Solutions							
PROBLEM						# of solutions circled	points
2	2a	2b	2c	2d	2e		
3	3a	3b	3c	3d	3e		
4	4a	4b	4c	4d	4e		
5	5a	5b	5c	5d	5e		
6	6a	6b	6c	6d	6e		

4.

$f^{(0)}(x) = x^{1/3}$	$f^{(0)}(8) = 2$	$c_0 = \frac{f^{(0)}(8)}{0!} = 2$
$f^{(1)}(x) = \frac{1}{3} x^{-2/3}$	$f^{(1)}(8) = \frac{1}{3} 2^{-2} = \frac{1}{12}$	$c_1 = \frac{f^{(1)}(8)}{1!} = \frac{1}{12}$
$f^{(2)}(x) = -\frac{1}{3} \cdot \frac{2}{3} x^{-5/3}$	$f^{(2)}(8) = -\frac{1}{9} \cdot \frac{2}{2^5} = -\frac{1}{9 \cdot 2^4}$	$c_2 = \frac{f^{(2)}(8)}{2!} = -1/(9)(2^5)$

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Not to Hand-in. Statement of Multiple Choice Problems.

2. Using a known (commonly used) Taylor series, find the Tayor series for

$$f(x) = \frac{1}{1+2x^3} = \frac{1}{1-(-2x^3)} \overset{GS}{=} \sum_{n=0}^{\infty} (-2x^3)^n$$

about the center $x_0 = 0$.

$$= \sum_{n=0}^{\infty} (-1 \cdot 2 \cdot x^3)^n$$

a. $\sum_{n=0}^{\infty} (-1)^n 2^n x^{3n}$ b. $\sum_{n=0}^{\infty} 2^n x^{3n}$
c. $\sum_{n=0}^{\infty} (-1)^n 2^n x^n$ d. $\sum_{n=0}^{\infty} 2^n x^n$
e. None of the others.

3. Find a power series reprsentation of

$$f(x) = \frac{x}{16+x^2} = \frac{x}{16} \frac{1}{\left(1 - -\frac{x^2}{16}\right)} \overset{GS}{=} \frac{x}{16} \sum_{n=0}^{\infty} \left(\frac{-x^2}{16}\right)^n$$

and state the LARGEST interval for which it is valid.

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{16} \cdot \frac{1}{16^n} x \cdot x^{2n}$$

a. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{16^{n+1}} x^{2n+1}$, valid precisely when $x \in (-4, 4)$
b. $f(x) = \sum_{n=0}^{\infty} \frac{1}{16^{n+1}} x^{2n+1}$, valid precisely when $x \in (-4, 4)$
c. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{16^{n+1}} x^{2n+1}$, valid precisely when $x \in (-16, 16)$
d. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{16^{n+1}} x^{2n+1}$, valid precisely when $x \in (-1, 1)$
e. None of the others.

valid when $\left|\frac{x^2}{16}\right| < 1$.

4. Find the 2^{nd} order Taylor polynomial for

$$f(x) = \sqrt[3]{x}$$

about the center $x_0 = 8$.

a. $2 + \frac{x}{12} - \frac{x^2}{9(2^5)}$ b. $2 + \frac{(x-8)}{12} + \frac{(x-8)^2}{9(2^5)}$ c. $2 + \frac{(x-8)}{12} - \frac{(x-8)^2}{9(2^5)}$ d. $2 + \frac{(x-8)}{12} - \frac{(x-8)^2}{9(2^4)}$
e. None of the others.

5. Suppose that the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is 9. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{2n}$?

$|x^2| < 9 \iff |x| < 3$

a. 1 b. 3 c. 9 d. 81 e. None of the others.

6. Find a power series representation of the function $y = f(t)$ where

$$f(t) = \int_0^t \frac{1}{1+x^7} dx = \int_0^t \frac{1}{1 - -x^7} dx =$$

and say for which values of t it is valid.

Geometric Series

a. $\sum_{n=0}^{\infty} \frac{t^{7n+1}}{7n+1}$, valid for $t \in (-1, 1)$ b. $\sum_{n=0}^{\infty} (-1)^n \frac{t^{7n+1}}{7n+1}$, valid for $t \in (-1, 1)$
c. $\sum_{n=1}^{\infty} \frac{t^{7n+1}}{7n+1}$, valid for $t \in (-1, 1)$ d. $\sum_{n=1}^{\infty} (-1)^n \frac{t^{7n+1}}{7n+1}$, valid for $t \in (-1, 1)$
e. None of the others.

$$= \int_0^t \left[\sum_{n=0}^{\infty} (-x^7)^n\right] dx = \sum_{n=0}^{\infty} (-1)^n \int_0^t x^{7n} dx = \sum_{n=0}^{\infty} (-1)^n \left.\frac{x^{7n+1}}{7n+1}\right|_{x=0}^{x=t}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{7n+1}}{7n+1}$$

valid (GS) when $|x^7| < 1$.

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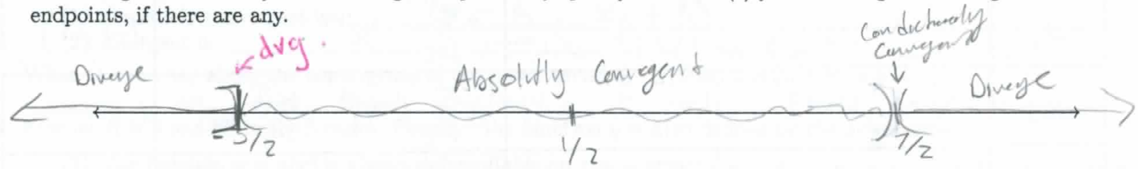
7. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n6^n}\ (2x-1)^n .$$

The center is $x_0 =$ $\frac{1}{2}$ and the radius of convergence is $R =$ 3 . As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



$\sum_{n=1}^{\infty} \frac{(-1^n)(2x-1)^n}{n6^n}$ Use Ratio test to make stuff Cancel

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|(2x-1)^{n+1}|}{(n+1)(6^{n+1})}\cdot\frac{n6^n}{|(2x-1)^n|} = \lim_{n\to\infty}\frac{|\cancel{(2x-1)^n}\cdot(2x-1)|\cdot n\cdot \cancel{6^n}}{(n+1)(\cancel{6^n})(6)|\cancel{(2x-1)^n}|}$$

the absolute value of -1^n is always 1 so we dont have to copy it over

$$= \lim_{n\to\infty}\frac{|2x-1|}{6}\cdot\frac{n}{n+1} = \frac{|2x-1|}{6}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|2x-1|}{6}\cdot 1$$

Since we are looking at n for the limit we can take everything without n outside the limit

Since we used the ratio test we knew $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ and $|L| < 1$ to converge so $\frac{|2x-1|}{6}$ has to be < 1 to converge

$$-1 < \frac{2x-1}{6} < 1$$

$$-6 < 2x-1 < 6$$

$$-3 < x - \tfrac{1}{2} < 3$$

$$-\tfrac{6}{2}+\tfrac{1}{2} < x < \tfrac{6}{2}+\tfrac{1}{2}$$

$$-\tfrac{5}{2} < x < \tfrac{7}{2}$$

endpts $-\frac{5}{2}$, and $\frac{7}{2}$

$\boxed{x_0 = \frac{1}{2} \quad R = 3}$

Check end point

$\sum_{n=1}^{\infty}\frac{(-1^n)(2x-1)^n}{n6^n}$ plug in $-\frac{5}{2}$ for x $\sum_{n=1}^{\infty}\frac{(-1)^n(2(-\frac{5}{2})-1)^n}{n6^n}$

$= \sum_{n=1}^{\infty}\frac{(-1)^n(-6)^n}{n6^n} = \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{6^n}{6^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ Diverges by p-series $p=1$

$\sum_{n=1}^{\infty}\frac{-1^n(2x-1)^n}{n6^n}$ plug in $\frac{7}{2}$ for x $\sum_{n=1}^{\infty}\frac{-1^n(2(\frac{7}{2})-1)^n}{n6^n} = \sum_{n=1}^{\infty}\frac{-1^n(6^n)}{n\cdot 6^n}$

$= \sum_{n=1}^{\infty}\frac{-1^n}{n}$ Conditionally converges by the alternating series test $\lim_{n\to\infty} C_n = 0$ ✓

C_n decreases as n get bigger ✓

Very Nice

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8. Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of the function

$$f(x) = e^{-5x}$$

about center $x_0 = 0$. Define the function remainder $y = R_N(x)$ by $R_N(x) := f(x) - P_N(x)$.

For each of **8a** and **8b**, below the answer box, you must work in a logical fashion, indicating your reasoning.

8a. For $|R_N(x)|$, find a good upper bound that Taylor's Remainder Theorem guarantees is valid for each $x \in \mathbb{R}$.
Hint. In your answer, you can have N and x and c but you need to indicate what you know about c.

Answer: $|R_N(x)| \le \left| \dfrac{5^{N+1} \cdot x^{N+1}}{(N+1)!\, e^{5c}} \right|$

c is a number between x and x_0 so...

$\underbrace{0 \le c \le x}_{\text{if } x \ge 0}$ or $\underbrace{x \le c \le 0}_{\text{if } x \le 0}$

Work:

$$|R_N(x)| = |f^{(N+1)}(c)| \cdot \frac{|x|^{N+1}}{(N+1)!}$$

$f(x) = e^{-5x}$

$f'(x) = -5e^{-5x}$

$f''(x) = +5^2 e^{-5x}$

$f^3(x) = -5^3 e^{-5x}$

$\vdots$

$f^{(n)}(x) = (-1)^n \cdot 5^n e^{-5x}$

$$|f^{(N+1)}(c)| = |(-1)^{N+1} \cdot 5^{N+1} \cdot e^{-5c}|$$

$$|R_N(x)| = \left| \frac{(-1)^{N+1} \cdot 5^{(N+1)} \cdot e^{-5c}}{(N+1)!} \right| \cdot |x|^{N+1} = \left| \frac{5^{N+1}}{e^{5c}(N+1)!} \right| \cdot |x|^{N+1}$$

$\boxed{\cancel{\underset{x_0}{0} \le c \le x}}$

c is a number between 0 & x

so if $x \ge 0$, then $0 \le c \le x$

and if $x \le 0$, then $x \le c \le 0$.

More space provided on (blank) next page.

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8b. For $|R_N(x)|$, find a good upper bound that Taylor's Remainder Theorem guarantees is valid for each $x \in (2, 3)$. Hint. In your answer, you can have N but cannot have x nor c.

Answer: $|R_N(x)| \leq \left| \dfrac{15^{N+1}}{(N+1)!} \right|$

Work:

From previous work on 8a:

$$|R_N(x)| \leq \left| \frac{5^{N+1} \cdot x^{N+1}}{(N+1)!\, e^{5c}} \right|$$

$$\leq \left| \frac{5^{N+1} \cdot 3^{N+1}}{(N+1)!\, e^{0}} \right| = \left| \frac{15^{N+1}}{(N+1)!} \right|$$

$e^0 = 1$

$2 \leq x \leq 3$

x is in numerator so need big number

$x = 3$

so...

$0 \leq c \leq 3$

|| multiply by 5

$0 \leq 5c \leq 15$

|| add e

$e^0 \leq e^{5c} \leq e^{15}$

c is in denominator so need small number to make a larger value

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