MARK BOX		
PROBLEM	POINTS POSSIBLE	YOUR SCORE
1–25	100	

NAME: solution key

PIN: ______

INSTRUCTIONS

- The MARK BOX above indicates the problems along with their points. Check that your copy of the exam has all of the problems.
- This exam comes in two parts.
  1. HAND IN PART. Hand in only this part, which includes a table for indicating your solutions to the multiple choice problems.
  2. STATEMENT OF MULTIPLE CHOICE PROBLEMS part. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.
- Upon request, you will be given as much (blank) scratch paper as you need.
- During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: electronic devices, books, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. Please, if I forget, remind me to pull up a clock on the projector screen.
- During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.
- This exam covers (from *Calculus* by Stewart, 6^{th} ed., ET): §7.1–7.5, 7.8, 11.1 – 11.11, 6.1 – 6.3, 10.3–10.4 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the above Instructions.

Signature : ______

- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to **2** answers for each problem. The scoring is as follows. For a problem with precisely one answer marked and the answer is correct, 4 points. For a problem with precisely two answers marked, one of which is correct, 1 points. All other cases, 0 points.
- Fill in the "number of solutions circled" column.

TABLE FOR YOUR ANSWERS							Do Not Write Below		
							points		
PROBLEM						number of solutions circled	4	1	0
1	**1a**	1b	1c	1d	1e				
2	2a	2b	2c	2d	**2e**				
3	3a	3b	3c	3d	**3e**				
4	4a	4b	**4c**	4d	4e				
5	**5a**	5b	5c	5d	5e				
6	6a	**6b**	6c	6d	6e				
7	7a	7b	7c	7d	**7e**				
8	8a	8b	8c	**8d**	8e				
9	9a	9b	9c	9d	**9e**				
10	10a	10b	10c	**10d**	10e				
11	11a	11b	**11c**	11d	11e				
12	12a	12b	12c	**12d**	12e				
13	13a	13b	**13c**	13d	13e				
14	14a	14b	**14c**	14d	14e				
15	**15a**	15b	15c	15d	15e				
16	16a	16b	**16c**	16d	16e				
17	17a	**17b**	17c	17d	17e				
18	18a	18b	18c	18d	**18e**				
19	**19a**	19b	19c	19d	19e				
20	**20a**	20b	20c	20d	20e				
21	21a	21b	**21c**	21d	21e				
22	22a	**22b**	22c	22d	22e				
23	**23a**	23b	23c	23d	23e				
24	24a	24b	24c	**24d**	24e				
25	25a	25b	25c	25d	25e				

- Hint for a typical (i.e. not improper) definite integral problems $\int_a^b f(x)\,dx$. First do the indefinite integral, say you get $\int f(x)\,dx = F(x) + C$. To check if you did this part correctly, you can use the Fundemental Theorem of Calculus (i.e. $F'(x)$ should be $f(x)$). Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.
- Hint: $\quad \ln b - \ln a = \ln\left(\frac{b}{a}\right)$ and $\ln(a^r) = r \ln a$ if $a, b > 0$ and $r \in \mathbb{R}$.

1. $\int\limits_{x=0}^{x=1} \frac{1}{x^2+1}\,dx = \arctan x|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

2. $\int\limits_{x=0}^{x=1} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int\limits_{x=0}^{x=1} \frac{2x\,dx}{x^2+1} = \frac{1}{2}\ln|x^2+1|\,|_{x=0}^{x=1} = \frac{1}{2}\ln(2) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(2) - 0 = \ln\sqrt{2}$.

3. Evaluate the integral $\int\limits_{x=0}^{x=e} \ln x\,dx$.

Note $\lim_{a\to 0^+} \ln a = -\infty$ and so we have an improper integral: $\int\limits_{x=0}^{x=e} \ln x\,dx = \lim_{a\to 0^+} \int\limits_{x=a}^{x=e} \ln x\,dx$.

EXAMPLE 2 Evaluate $\int \ln x\,dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x}\,dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x\,dx = x\ln x - \int x\,\frac{dx}{x}$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f.

If $a > 0$, then $\int\limits_a^e \ln x\,dx = (x\ln x - x)\,|_a^e = (e\ln e - e) - (a\ln a - a) = -a\ln a + a = a(1 - \ln a)$.

So $\int\limits_0^e \ln x\,dx = \lim_{a\to 0^+} \int\limits_a^e \ln x\,dx = \lim_{a\to 0^+} a(1-\ln a) \overset{(0)(\infty)}{=} \lim_{a\to 0^+} \frac{1-\ln a}{a^{-1}} \overset{\text{L'H}}{=} \lim_{a\to 0^+} \frac{-1/a}{-1/a^2} = \lim_{a\to 0^+} a = 0$.

4. $\displaystyle\int_{x=0}^{x=\frac{\pi}{2}} \cos^3 x \sin^4 x \, dx = \left(\tfrac{1}{5} - \tfrac{1}{7}\right) - (0 - 0) = \tfrac{2}{35}$.

Example 4 Evaluate $\int \cos^3 x \sin^4 x \, dx$.

Solution We proceed as follows:

$$\int \cos^3 x \sin^4 x \, dx = \int \cos^2 x \sin^4 x \cos x \, dx$$
$$= \int (1 - \sin^2 x) \sin^4 x \cos x \, dx.$$

If we let $u = \sin x$, then $du = \cos x \, dx$ and the integral may be written

$$\int \cos^3 x \sin^4 x \, dx = \int (1 - u^2) u^4 \, du$$
$$= \int (u^4 - u^6) \, du$$
$$= \tfrac{1}{5} u^5 - \tfrac{1}{7} u^7 + C$$
$$= \tfrac{1}{5} \sin^5 x - \tfrac{1}{7} \sin^7 x + C.$$

5. $\displaystyle\int_2^3 \frac{4x^2+13x-9}{x^3+2x^2-3x} \, dx = \ln \left| \frac{x^3(x-1)^2}{x+3} \right| \Big|_2^3 = \ln \left| \frac{3^3 2^2}{3 \cdot 2} \cdot \frac{5}{2^3 \cdot 1^2} \right| = \ln \frac{3^2 5}{2^2} = \ln \frac{45}{5}$

Example 1 Evaluate $\displaystyle\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} \, dx.$

Solution The denominator of the integrand has the factored form $x(x + 3)(x - 1)$. Each of the linear factors is handled under Rule 1, with $m = 1$. Thus, for the factor x there corresponds a partial fraction of the form A/x. Similarly, for the factors $x + 3$ and $x - 1$ there correspond partial fractions $B/(x + 3)$ and $C/(x - 1)$, respectively. The decomposition (10.3) then has the form

$$\frac{4x^2 + 13x - 9}{x(x + 3)(x - 1)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{x - 1}.$$

Multiplying by the lowest common denominator gives us

$$4x^2 + 13x - 9 = A(x + 3)(x - 1) + Bx(x - 1) + Cx(x + 3). \tag{10.4}$$

Multiplying by the lowest common denominator gives us

(10.4) $$4x^2 + 13x - 9 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3).$$

In a case such as this, in which the factors are all linear and nonrepeated, the values for A, B, and C can be found by substituting values for x which make the various factors zero. If we let $x = 0$ in (10.4), then

$$-9 = -3A \quad \text{or} \quad A = 3.$$

Letting $x = 1$ in (10.4) gives us

$$8 = 4C \quad \text{or} \quad C = 2.$$

Finally, if $x = -3$, then

$$-12 = 12B \quad \text{or} \quad B = -1.$$

The partial fraction decomposition is, therefore,

$$\frac{4x^2 + 13x - 9}{x(x+3)(x-1)} = \frac{3}{x} + \frac{-1}{x+3} + \frac{2}{x-1}.$$

Integrating,

$$\begin{aligned}\int \frac{4x^2 + 13x - 9}{x(x+3)(x-1)}\,dx &= \int \frac{3}{x}\,dx + \int \frac{-1}{x+3}\,dx + \int \frac{2}{x-1}\,dx \\ &= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + D \\ &= \ln|x^3| - \ln|x+3| + \ln|x-1|^2 + D \\ &= \ln\left|\frac{x^3(x-1)^2}{x+3}\right| + D.\end{aligned}$$

Another technique for finding A, B, and C is to compare coefficients of x. If the right-hand side of (10.4) is expanded and like powers of x are collected, then

$$4x^2 + 13x - 9 = (A + B + C)x^2 + (2A - B + 3C)x - 3A.$$

We now use the fact that if two polynomials are equal, then coefficients of like powers are the same. Thus

$$\begin{cases} A + B + C = 4 \\ 2A - B + 3C = 13 \\ -3A = -9. \end{cases}$$

It is left to the reader to show that the solution of this system of equations is $A = 3$, $B = -1$, and $C = 2$.

6. $\int_0^1 \frac{1}{\sqrt{x^2+8x+25}}\,dx = \ln\left|\sqrt{x^2+8x+25} + x + 4\right| \Big|_0^1 = \ln\left|\sqrt{34}+5\right| - \ln\left|\sqrt{25}+4\right| = \ln\frac{\sqrt{34}+5}{9}$

Hint: $x^2 + 8x + 25 = (x+4)^2 + 9$.

Example 3 Evaluate $\int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx$.

Solution We complete the square for the quadratic expression as follows:

$$\begin{aligned} x^2 + 8x + 25 &= (x^2 + 8x \qquad) + 25 \\ &= (x^2 + 8x + 16) + 25 - 16 \\ &= (x + 4)^2 + 9. \end{aligned}$$

Hence

$$\int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx = \int \frac{1}{\sqrt{(x + 4)^2 + 9}}\,dx.$$

If we next make the trigonometric substitution

$$x + 4 = 3\tan\theta$$

then

$$dx = 3\sec^2\theta\,d\theta$$

$$\sqrt{(x + 4)^2 + 9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\tan^2\theta + 1} = 3\sec\theta$$

and hence

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx &= \int \frac{1}{3\sec\theta} 3\sec^2\theta\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C. \end{aligned}$$

In order to return to the variable x we use the triangle in Figure 10.4. This gives us

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx &= \ln\left|\frac{\sqrt{x^2 + 8x + 25}}{3} + \frac{x + 4}{3}\right| + C \\ &= \ln|\sqrt{x^2 + 8x + 25} + x + 4| + D \end{aligned}$$

where $D = C - \ln 3$.

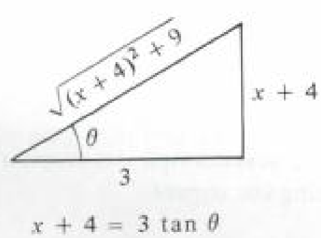


Figure 10.4

7. $\int_{x=0}^{x=\frac{\pi}{2}} \cos^4 x \; dx = \left(\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right)\Big|_0^{\pi/2} = \left(\frac{3\pi}{16} - 0 - 0\right) - (0 - 0 + 0) = \frac{3\pi}{16}$.

Example 4 Two successive applications of the half-angle formula for the cosine give

$$\cos^4 x = (\cos^2 x)^2 = \tfrac{1}{4}(1 + \cos 2x)^2 = \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$
$$= \tfrac{1}{4}[1 + 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)]$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x,$$

so

$$\int \cos^4 x \, dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + c.$$

8. $\int_{x=0}^{x=\frac{3\pi}{2}} e^x \cos x \; dx = \frac{e^x(\sin x + \cos x)}{2}\Big|_0^{3\pi/2} = \frac{e^{3\pi/2}(-1)}{2} - \frac{e^0(1)}{2} = \frac{-1-e^{3\pi/2}}{2}$

Example 5 Find $\int e^x \cos x \, dx$.

Solution Let

$$u = e^x \qquad dv = \cos x \, dx$$
$$du = e^x \, dx \qquad v = \sin x.$$

Integrating by parts,

(a) $$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

We next apply integration by parts to the integral on the right side of equation (a). Letting

$$u = e^x \qquad dv = \sin x \, dx$$
$$du = e^x \, dx \qquad v = -\cos x$$

and integrating by parts leads to

(b) $$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.$$

If we now use equation (b) to substitute on the right side of equation (a) we obtain

$$\int e^x \cos x \, dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx\right]$$

or

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

Adding $\int e^x \cos x \, dx$ to both sides gives us

$$2\int e^x \cos x \, dx = e^x(\sin x + \cos x).$$

Finally, dividing both sides by 2 and adding the constant of integration, we have

$$\int e^x \cos x \, dx = \tfrac{1}{2}e^x(\sin x + \cos x) + C.$$

9. $\int_{x=-1}^{x=1} \frac{1}{x^{2/3}}\, dx = \lim_{a\to 0^-} \int_{x=-1}^{x=a} \frac{1}{x^{2/3}}\, dx + \lim_{b\to 0^+} \int_{x=b}^{x=1} \frac{1}{x^{2/3}}\, dx \overset{\text{see below}}{=} 3+3=6.$

Note that $\int_0^1 x^{\frac{-2}{3}}\, dx = \lim_{b\to 0^+} \int_b^1 x^{-\frac{2}{3}}\, dx = \lim_{b\to 0^+} 3x^{\frac{1}{3}} \mid_b^1 = \lim_{b\to 0^+} \left(3 - 3b^{\frac{1}{3}}\right) = 3.$

Similiarly (also by symmetry) $\lim_{a\to 0^-} \int_{-1}^{a} x^{\frac{-2}{3}}\, dx = 3.$

10. $\int_{x=-1}^{x=1} \frac{1}{x^3}\, dx$ does not exist but also does not diverge to infinity.

• $\int x^{-3}\, dx = \frac{x^{-2}}{-2} + C$

$\int_{x=0}^{x=1} x^{-3}\, dx = \lim_{a\to 0^+} \frac{x^{-2}}{-2} \Big|_{x=a}^{x=1} = \frac{1}{2} \lim_{a\to 0^+} \left[\frac{1}{x^2}\right]_{x=1}^{x=a} =$

$\frac{1}{2} \lim_{x\to 0^+} \left[\frac{1}{a^2} - 1\right] = \infty.$ Similarly, $\int_{-1}^{0} x^{-3}\, dx = -\infty.$

• $\int_{-1}^{1} x^{-3}\, dx = \int_{-1}^{0} x^{-3}\, dx + \int_b^1 x^{-3}\, dx = -\infty + \infty$ so DNE.

11. $\frac{\sqrt{25n^8+5n^7-n^2+1}}{3n^4+5n^2-n-2} = \frac{\sqrt{25n^8+5n^7-n^2+1}}{3n^4+5n^2-n-2} \cdot \frac{\frac{1}{\sqrt{n^8}}}{\frac{1}{n^4}} = \frac{\sqrt{25+5n^{-1}-n^{-6}+n^{-8}}}{3+5n^{-2}-n^{-3}-2n^{-4}} \xrightarrow{n\to\infty} \frac{5}{3}$.

12. For what value $r \in R$ does $\sum_{n=2}^{\infty} r^n = \frac{1}{4}$?Answer: when $r = \frac{-1\pm\sqrt{17}}{8}$.

Let $s_N = \sum_2^N r^n$. Then

$$s_N = r^2 + r^3 + \ldots + r^N$$
$$rs_N = r^3 + r^4 + \ldots + r^{N+1}.$$

So if $|r| < 1$, then

$$s_N = \frac{r^2 - r^{N+1}}{1-r} \xrightarrow{N\to\infty} \frac{r^2}{1-r} .$$

So we want $\frac{r^2}{1-r} = \frac{1}{4}$. And

$$\frac{r^2}{1-r} = \frac{1}{4} \Leftrightarrow 4r^2 = 1-r \Leftrightarrow 4r^2+r-1=0 \Leftrightarrow r = \frac{-1\pm\sqrt{1-(4)(4)(-1)}}{8} \Leftrightarrow r = \frac{-1\pm\sqrt{17}}{8} .$$

Note $4 = \sqrt{16} < \sqrt{17} < \sqrt{25} = 5$ and so

$$0 < \frac{3}{8} < \frac{-1+\sqrt{17}}{8} < \frac{4}{8} < 1 \quad \text{and} \quad -1 < \frac{-6}{8} < \frac{-1-\sqrt{17}}{8} < \frac{-5}{8} < 0$$

and so $\left|\frac{-1\pm\sqrt{17}}{8}\right| < 1.$

13. The formal series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}}$ converges conditionally as can be shown by using the LCT with $b_n = \frac{1}{n}$ as well as the AST.

Problem Inspiration: 09 Fall final #5

Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)\,(n+7)}}$$

- ☐ ~~absolutely convergent~~ LCT w/ $b_n = \frac{1}{n}$
- ☒ conditionally convergent — then use AST
- ☐ divergent

Warning: there is a square root in the denominator ... many of you overlooked $\sqrt{\ }$'s on Exams ... see it?.

Abs. Conv? Consider $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right| = \sum \frac{1}{\sqrt{(n+2)(n+7)}}$

Thinking Land: $a_n = \frac{1}{\sqrt{(n+2)(n+7)}} \overset{n \text{ big}}{\approx} \frac{1}{\sqrt{n \cdot n}} = \frac{1}{n} = b_n$

LCT. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{(n+2)(n+7)}} = 1$

more details → $= \lim_{n\to\infty} \sqrt{\left[\frac{n^2}{(n+2)(n+7)}\right]} = \sqrt{1} = 1 = L$, $0 < L < \infty$

so $\sum b_n$ & $\sum a_n$ do the same thing. $\sum b_n$ divg (harmonic series)

so $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right|$ dvg so not abs. conv.

Cond. Conv? Let's use AST w/ $0 \le u_n = \frac{1}{\sqrt{(n+2)(n+7)}}$

(1) u_n dec., ie $u_n > u_{n+1}$? yes clear.

(2) $\lim_{n\to\infty} \frac{1}{\sqrt{(n+2)(n+7)}} = 0$ ☺

} so, by AST, $\sum (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}}$ conv.

10

14. $\sum_{n=2}^{\infty} \frac{1}{n\ (\ln n)^2}$ converges, as can be shown by the integral test.

. The function $f(x) = \dfrac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{\ln x}\right]_2^t \quad \text{[by substitution with } u=\ln x\text{]} \quad = -\lim_{t\to\infty}\left(\frac{1}{\ln t}-\frac{1}{\ln 2}\right)=\frac{1}{\ln 2},$$

so the series $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^2}$ converges.

15. $\sum_{n=2}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$converges by the Root Test.

EXAMPLE 6 Test the convergence of the series $\sum_{n=1}^{\infty}\left(\dfrac{2n+3}{3n+2}\right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

16. Let c be a natural number (i.e., $c \in \{1, 2, 3, 4, \ldots\}$).
The series $\sum_{n=1}^{\infty} \frac{(n!)^6}{(cn)!}\frac{(n!)^6}{(cn)!}$ diverges when $c < 6$ and converges when $c \geq 6$.
Let $a_n = \frac{(n!)^6}{(cn)!}$ and $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{[(n+1)!]^6}{[n!]^6}\ \frac{(cn)!}{(cn+c)!} = \frac{(n+1)^6}{(cn+1)(cn+2)\cdots(cn+c)}$$
$$= \frac{n^6 + \text{(a poly. of degree atmost 5)}}{c^c\,(n^c) + \text{(a poly. of degree at most (c-1))}}\ .$$

If $c < 6$, then $\rho = \infty$. If $c > 6$, then $\rho = 0$, If $c = 6$, then $\rho = \frac{1}{6^6} < 1$. Now apply the ratio test.

17. What is the LARGEST set for which the formal power series $\sum_{n=17}^{\infty} \frac{x^n}{n!}$is convergent (either absolutely or conditionally , so, in other words, its interval of convergence)? Answer: $(-\infty, +\infty)$

Let $u_k = \frac{x^k}{k!}$. Applying the ratio test for absolute convergence, we obtain

$$\rho = \lim_{k\to+\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to+\infty}\left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right| = \lim_{k\to+\infty}\left|\frac{x}{k+1}\right| = 0$$

Since $\rho < 1$ for all x, the series converges absolutely for all x. Thus, the interval of convergence is $(-\infty, +\infty)$ and the radius of convergence is $R = +\infty$.

18. We know that the geometric series

$$\sum_{n=0}^{\infty} r^n \quad \begin{cases} = \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \; . \end{cases}$$

So if $|2x^3| < 1$, i.e. if $x \in \left(-2^{-1/3}, 2^{-1/3}\right)$, then $\frac{1}{1-2x^3} = \sum_{n=0}^{\infty} (2x^3)^n = \sum_{n=0}^{\infty} 2^n\, x^{3n}$.

19. We know that the geometric series

$$\sum_{n=0}^{\infty} r^n \quad \begin{cases} = \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \; . \end{cases}$$

So $\frac{1}{3-2x} = \frac{1}{1-2(x-1)} = \sum_{n=0}^{\infty} (2\,(x-1))^n = \sum_{n=0}^{\infty}\; 2^n (x-1)^n$

This expansion is valid when $|2(x-1)| < 1$, i.e., when $0.5 < x < 1.5$.

20. The 3rd order Taylor polynomial for $f(x) = \frac{1}{x}$ about the center $x_0 = 2$ is

$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-1}	$1/2$
1	$-x^{-2}$	$-1/4$
2	$2x^{-3}$	$+1/4$
3	$-6x^{-4}$	$-3/8$

$$P_3(x) = \frac{f^{(0)}(2)}{0!} + \frac{f^{(1)}(2)}{1!}(x-2)^1 + \frac{f^{(2)}(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}\cdot\frac{1}{2}(x-2)^2 - \frac{3}{8}\cdot\frac{1}{2\cdot 3}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

21. Answer: $|R_5(x)| \leq \frac{e(3^6)}{6!}$

For each $x \in (-1,3)$, there exists $c \in (-1,3)$ so that

$$|R_5(x)| = \left| \frac{f^{(6)}(c)}{6!}(x-0)^6 \right| = \frac{1}{6!} e^{-c} |x|^6 \leq \frac{1}{6!} e^{-(-1)}\, 3^6$$

22. Answer: $x^2 + (y-1)^2 = 1$

$r = 2\sin\theta \iff r^2 = 2r\sin\theta$

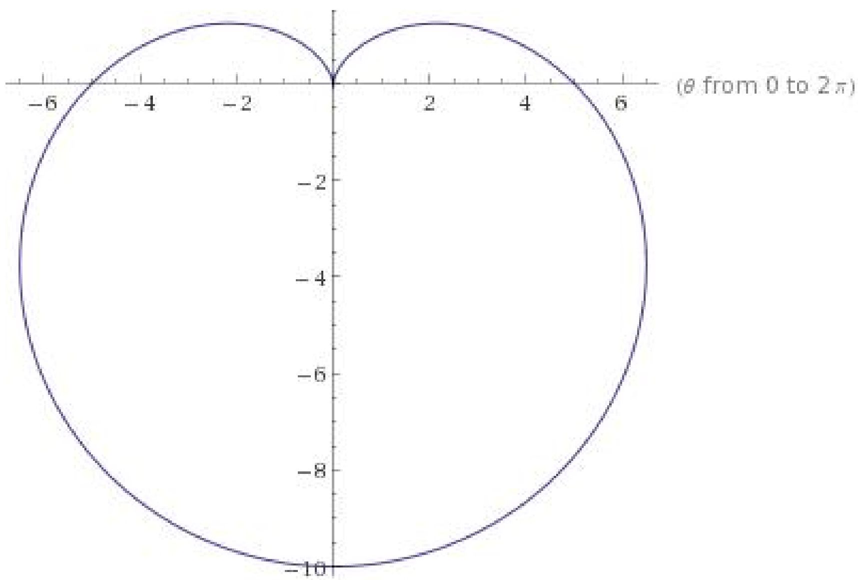
$\iff x^2 + y^2 = 2y$ ← ok, but can you do better? what is it?

$\iff x^2 + y^2 - 2y + 1 = 1$

$\iff x^2 + (y-1)^2 = 1.$ ← circle with { radius = 1, center = (0, 1) }

23. The area enclosed by $r = 5 - 5\sin\theta$ is $\frac{1}{2}\int_0^{2\pi}[5 - 5\sin\theta]^2\ d\theta$
$\frac{1}{4}$ (period) $= \frac{1}{4}\frac{2\pi}{1} = \frac{\pi}{2}$.

θ	$5\sin\theta$	$r = 5 - 5\sin\theta$
$0 \to \frac{\pi}{2}$	$0 \to 5$	$5 \to 0$
$\frac{\pi}{2} \to \pi$	$5 \to 0$	$0 \to 5$
$\pi \to \frac{3\pi}{2}$	$0 \to -5$	$5 \to 10$
$\frac{3\pi}{2} \to 2\pi$	$-5 \to 0$	$10 \to 5$



Now consider a function $r = f(\theta)$ which determines a curve in the plane where

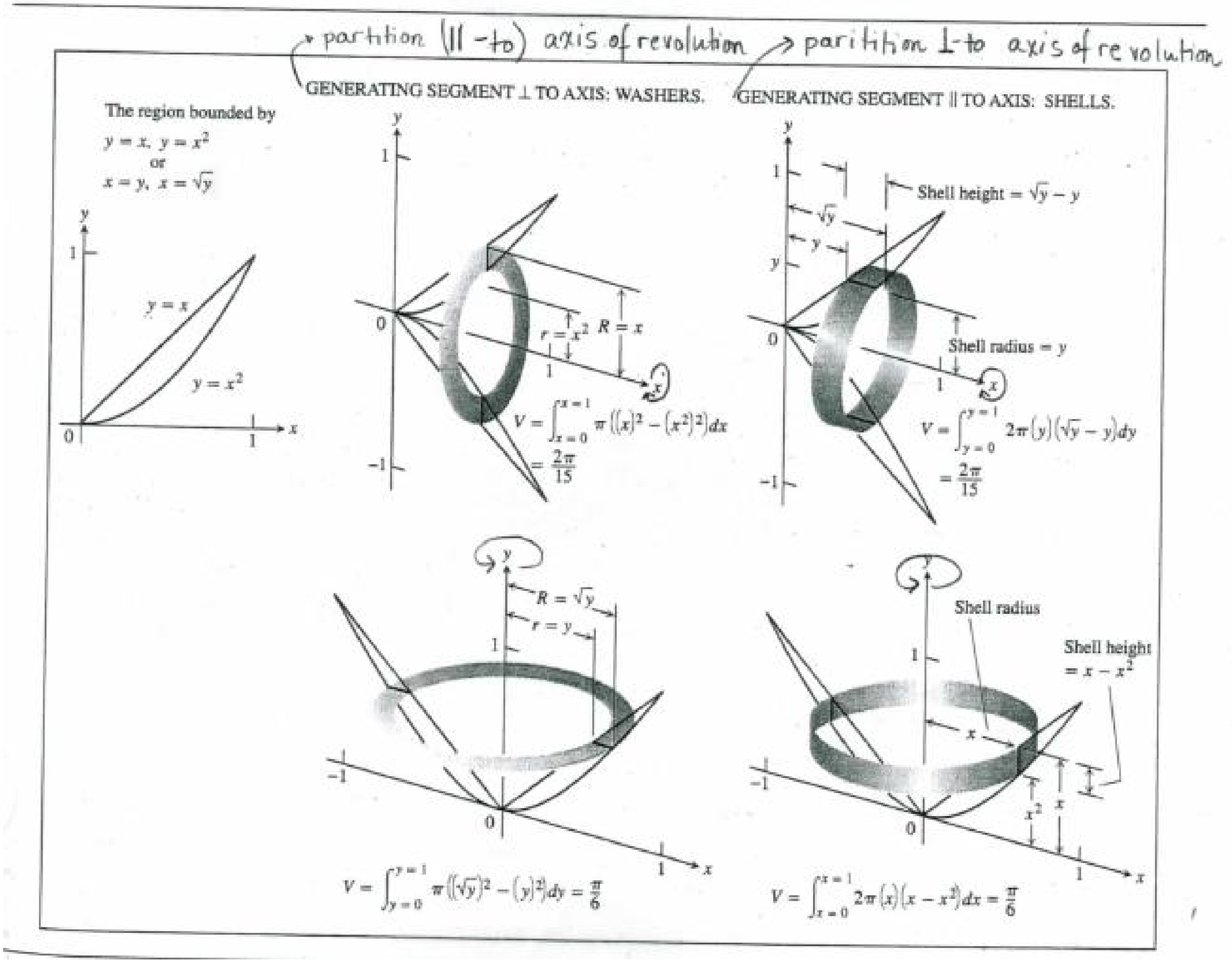
(1) $f\colon [\alpha, \beta] \ \to\ [0, \infty]$
(2) f is continuous on $[\alpha, \beta]$
(3) $\beta - \alpha \le 2\pi$.

Then the area bounded by polar curves $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A \ = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\ [f(\theta)]^2 \ d\theta\ .$$

24. The volume of V is $2\pi \int_{x=0}^{x=1} (y)\left(\sqrt{y} - y\right)\, dy$

Table 5.1 Washers vs. shells



25. Of course, my answer is 17. What is yours?