1. From 13 Spring, Final Exam Review (posted under Previous Exams).

- Ex: $\int xe^x\,dx$

$u = x \quad dv = e^x\,dx$

$du = dx \quad v = e^x$ gives

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C. = (x-1)e^x + C$$

- Check $D_x[(x-1)e^x] = [D_x(x-1)]e^x + (x-1)[D_x e^x] = 1e^x + (x-1)e^x = xe^x.$ ✓

- $\int_{x=0}^{x=1} xe^x\,dx = (x-1)e^x \Big|_{x=0}^{x=1} = [0-(-e^0)] = e^0 = 1.$

2. • First, Find $\int \arcsin x\,dx$.

$\boxed{\text{parts}}$, because $y=\sin^{-1}x$ is easy to differentiate but hard to integrate so try parts with

Solution We make the substitutions

$$u = \arcsin x \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$$

Use parts for same reason use parts for $\int \ln x\,dx$.

Then

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arcsin x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x\,dx)$$

$$= x\arcsin x + \frac{1}{2}\cdot 2(1-x^2)^{1/2} + C$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

- Check $[D_x(x\sin^{-1}x + (1-x^2)^{1/2})] = 1\cdot\sin^{-1}x + x(1-x^2)^{-1/2} + \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \sin^{-1}x$

- $\int_{x=0}^{x=1} \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_{x=0}^{x=1} = [\sin^{-1}1 + 0] - [0+1] = \frac{\pi}{2} - 1.$

3. • From class lecture handout on Trig Integrals

Example 4. $\int \sin^4 x\,dx$

If we try $s = \cos x$ or $t = \sin x$, it will not work (why? $\int \sin^4 x\,dx = -\int \boxed{\sin^3 x}\,[-\sin x\,dx]$).
Here we use the half-angle formulas.

$$\int \sin^4 x\,dx = \int [\sin^2 x]^2\,dx = \int \left[\frac{1-\cos(2x)}{2}\right]^2 dx = \frac{1}{4}\int [1 - 2\cos(2x) + \cos^2(2x)]\,dx$$
$$= \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] dx$$
$$= \frac{3}{8}\int dx - \frac{1}{4}\cdot\int \cos(2x)\,2dx + \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}\int \cos(4x)\,4dx$$
$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

• Check: use half/double-angle formulas.

$$D_x\left[\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right]$$
$$= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$
$$= \left[\frac{1-\cos(2x)}{2}\right] + \frac{1}{4}\left[\frac{1+\cos 2(2x)}{2}\right] + -\frac{1}{4}$$
$$= \sin^2(x) + \frac{1}{4}\cos^2(2x) - \frac{1}{4} = \sin^2(x) + \frac{1}{4}\Big[\underbrace{\cos^2(x) - \sin^2(x)}_{1-\sin^2(x)}\Big]^2 - \frac{1}{4}$$
$$= \sin^2(x) + \frac{1}{4}\left[1 - 2\sin^2(x)\right]^2 - \frac{1}{4}$$
$$= \sin^2(x) + \frac{1}{4}\left[1 - 4\sin^2(x) + 4\sin^4(x)\right] - \frac{1}{4}$$
$$= \sin^2(x) + \frac{1}{4} - \sin^2(x) + \sin^4(x) - \frac{1}{4} = \sin^4(x)\ \checkmark$$

• $\int_0^1 \sin^4 x\,dx = \left[\frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32}\right]\Big|_{x=0}^{x=1} = \left[\frac{3}{8} - \frac{1}{4}\sin 2 + \frac{1}{32}\sin 4\right] - [0]$

4. • First, Evaluate $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$, assuming that $x \geq 5$.

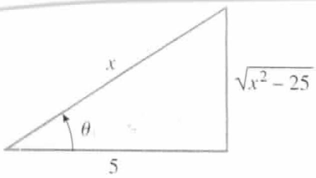
Solution. The integrand involves a radical of the form $\sqrt{x^2 - a^2}$ with $a = 5$, so from Table 8.4.1 we make the substitution

$$x = 5\sec\theta, \quad 0 \leq \theta < \pi/2$$
$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \text{or} \quad dx = 5\sec\theta\tan\theta\,d\theta$$

2014S, Final, p3

Thus,

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$
$$= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$
$$= 5\int \tan^2\theta\,d\theta \qquad \tan\theta \ge 0 \text{ since } 0 \le \theta < \pi/2$$
$$= 5\int (\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C$$



$x = 5\sec\theta$

Figure 8.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 8.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C \blacktriangleleft$$

• Check $D_x\left[(x^2-25)^{1/2} - 5\sec^{-1}\left(\frac{x}{5}\right)\right]$

$$= \frac{1}{2}(x^2-25)^{-1/2}(2x) - 5\,\frac{1}{\left|\frac{x}{5}\right|\sqrt{\left(\frac{x}{5}\right)^2-1}}\cdot\frac{1}{5}$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{1}{\frac{x}{5}\sqrt{\frac{x^2}{25}-\frac{25}{25}}} \qquad (\text{know } x \ge 5)$$

$$= \frac{x}{(x^2-25)^{1/2}} - \frac{25}{x(x^2-25)^{1/2}} = \frac{(x^2-25)}{x(x^2-25)^{1/2}} \cdot \frac{(x^2-25)^{1/2}}{(x^2-25)^{1/2}}$$

$$= \frac{(x^2-25)(x^2-25)^{1/2}}{x(x^2-25)} = \frac{\sqrt{x^2-25}}{x}. \checkmark$$

• $$\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)\Big|_{x=5}^{x=10}$$

$$= \left[\sqrt{100-25} - 5\sec^{-1}2\right] - \left[0 - \cancelto{0}{5\sec^{-1}1}\right] =$$

$$= \sqrt{75} - 5\cdot\frac{\pi}{3} = 5\sqrt{3} - 5\left(\frac{\pi}{3}\right) = 5\left(\sqrt{3} - \frac{\pi}{3}\right).$$

5. • $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

• Check $D_x\left[2\ln|x| + x^{-1} + 3\ln|x+2|\right] = \frac{2}{x} + -1x^{-2} + \frac{3}{x+2}$

$$= \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2} = \frac{2x(x+2) - (x+2) + 3x^2}{x^2(x+2)} = \frac{5x^2+3x-2}{x^3+2x^2} \checkmark$$

• $\left[3\ln|x+2| + 2\ln x + \frac{1}{x}\right]\Big|_{x=1}^{x=3} =$

$$\left[3\ln 5 + 2\ln 3 + \frac{1}{3}\right] - \left[3\ln 3 + \underbrace{2\ln 1}_{=0} + 1\right] =$$

$$3\ln 5 - \ln 3 - \frac{2}{3}.$$

6. • $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$

• $\int_{x=0}^{x=1} x^{-3}\,dx = \lim_{a\to0^+} \frac{x^{-2}}{-2}\Big|_{x=a}^{x=1} = \frac{1}{2}\lim_{a\to0^+}\left[\frac{1}{x^2}\right]_{x=1}^{x=a} =$

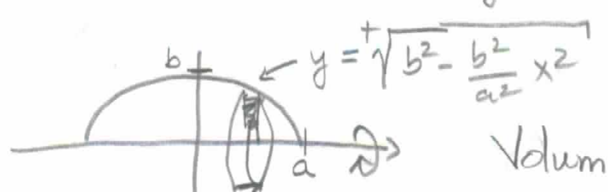
$\frac{1}{2}\lim_{x\to0^+}\left[\frac{1}{a^2} - 1\right] = \infty$. Similarly, $\int_{-1}^{0} x^{-3}\,dx = -\infty$.

• $\int_{-1}^{1} x^{-3}\,dx = \int_{-1}^{0} x^{-3}\,dx + \int_{0}^{1} x^{-3}\,dx = -\infty + \infty$ so DNE.

7. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \Leftrightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Leftrightarrow y^2 = b^2 - \frac{b^2}{a^2}x^2$

$$\Leftrightarrow y = \pm\sqrt{b^2 - \frac{b^2}{a^2}x^2}$$

Note, the integral is easy to compute, it's $\frac{4}{3}\pi ab^2$. If $a=b$, get a circle w/ radius $r=a=b$, with, rotate to get sphere w/ $V = \frac{4}{3}\pi r^3$



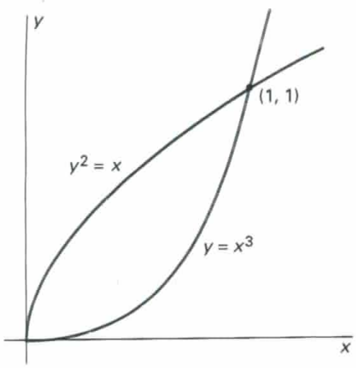
Volume of typical disk $= \pi r^2 h$

$$= \pi\left(\sqrt{b^2 - \frac{b^2}{a^2}x^2}\right)^2 \Delta x$$

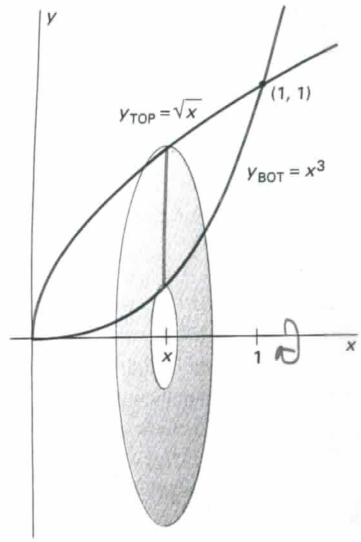
$\boxed{\text{Disk Method}}$

$$V = \pi\int_{-a}^{a}\left(b^2 - \frac{b^2}{a^2}x^2\right)dx \overset{\text{symmetry}}{=} 2\pi\int_{0}^{a}\left(b^2 - \frac{b^2}{a^2}x^2\right)dx$$

8. **EXAMPLE 4** Consider the plane region shown in Fig. 6.14, bounded by the curves $y^2 = x$ and $y = x^3$, which intersect at the points (0, 0) and (1, 1). If this



6.14 The plane region of Example 4



6.15 Revolution about the x-axis

Washer Method

region is revolved about the x-axis (Fig. 6.15), then the formula in (6) with

radius big $= y_{\text{top}} = \sqrt{x}, \qquad y_{\text{bot}} = x^3$ = radius little

gives

$$V = \int_0^1 \pi[(\sqrt{x})^2 - (x^3)^2]\,dx = \int_0^1 \pi(x - x^6)\,dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{7}x^7\right]_0^1 = \tfrac{5}{14}\pi$$

9. Find the volume of the solid obtained by revolving about the y axis the region in the first quadrant bounded above by the parabola $y = 2 - x^2$ and below by the parabola $y = x^2$. (See Fig. 30-12.)

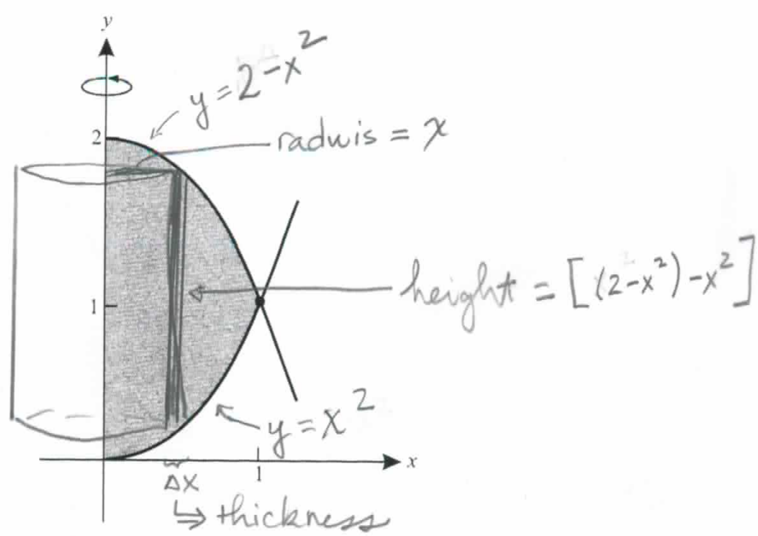
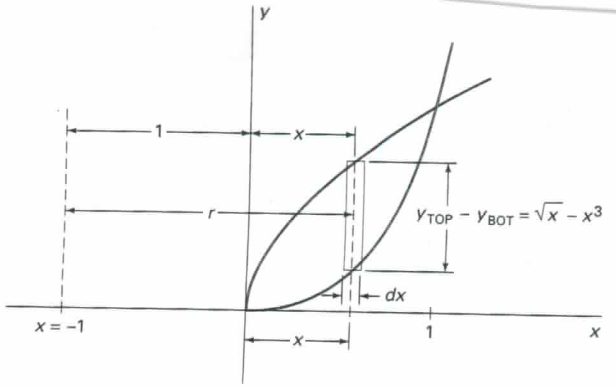


Fig. 30-12

The curves intersect at (1,1). By the cylindrical shells formula, the volume is

$$V = 2\pi \int_0^1 \underbrace{x}_{\text{radius}}\underbrace{((2-x^2)-x^2)}_{\text{height}}\,\underbrace{dx}_{\text{thickness}} = 4\pi \int_0^1 (x - x^3)\,dx = 4\pi\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big]_0^1 = 4\pi\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \pi$$

10.



= height

6.36 Revolution about the line $x = -1$

is revolved through a circle of radius $r = 1 + x$. Hence the volume of the resulting cylindrical shell is

$$dV == 2\pi r\, dA = 2\pi\overbrace{(1+x)}^{\text{radius}}\overbrace{(x^{1/2} - x^3)}^{\text{height}}\overbrace{dx}^{\text{thickness}}$$
$$= 2\pi(x^{1/2} + x^{3/2} - x^3 - x^4)\, dx,$$

so the volume of the resulting solid of revolution is

$$V = \int_0^1 2\pi(x^{1/2} + x^{3/2} - x^3 - x^4)\, dx$$
$$= 2\pi\left[\tfrac{2}{3}x^{3/2} + \tfrac{2}{5}x^{5/2} - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^1 = \tfrac{37}{30}\pi,$$

11. The ball goes down 30 feet, up $(5/6)30$ feet, down $(5/6)30$ feet, up $(5/6)^2 30$ feet, down $(5/6)^2 30$ feet, up $(5/6)^3 30$ feet, down $(5/6)^3 30$ feet, etc. The total distance travelled is

$$30 + \left((5/6)60 + (5/6)^2 60 + (5/6)^3 60 + (5/6)^4 60 + \dots\right) \text{ feet.}$$

The sum inside the big parentheses is the geometric series with initial term $a = (5/6)60$ and ratio $r = 5/6$. We see that $-1 < r < 1$, so the geometric series converges to $\boxed{\frac{a}{1-r}}$. Thus the total distance travelled is

$$\boxed{30 + \frac{(5/6)60}{1 - 5/6} \text{ feet.}} = 330 \text{ feet.}$$

you should be able to derive this formula!

12. 12. $\frac{2}{k^2-1}$ $\underline{\text{PDF}}$ $\frac{A}{k-1} + \frac{B}{k+1}$ $\underline{\text{solve it}}$ $\frac{1}{k-1} - \frac{1}{k+1}$

So $s_n = \sum_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k+1}$.

This sum is equal to

$$s_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots + \left(\frac{1}{n-4} - \frac{1}{n-2}\right)$$
$$+ \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

The $1/3$'s cancel. The $1/4$'s cancel. The $-1/5$ cancels with the left most $+1/5$ from the first unwritten term. The $-1/6$ cancels with the left most $1/6$ of the second unwritten term., etc. Now look at the right side. The $\frac{1}{n-1}$'s cancel. The $\frac{1}{n-2}$'s cancel. The $\frac{1}{n-3}$ cancels with the right most $-\frac{1}{n-3}$ from the last unwritten term. The $\frac{1}{n-4}$ also cancels with a term in the middle. We are left with

$$\boxed{s_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}.} \xrightarrow{n \to \infty} \frac{3}{2} = s.$$

13. Series A, $\sum \frac{(-1)^n}{n}$, is CC since:

- $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$. dvg. p-series, $p = 1 \leq 1$
- $\sum \frac{(-1)^n}{n}$ conv by AST since $\frac{1}{n} \xrightarrow{n \to \infty} 0$.

Series B, $\sum \frac{(-1)^n}{n^2}$, is AC since:

- $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ conv., p-series, $p = 2 > 1$.

14. Thinking land: $u_n = \frac{1}{\sqrt[3]{(n+2)(n+5)(n+7)}} \overset{n \text{ big}}{\approx} \frac{1}{\sqrt[3]{n \cdot n \cdot n}} = \frac{1}{n} = b_n$

Note $u_n > 0$.

Series A is DVG by LCT using $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{(n+2)(n+5)(n+7)}} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \left[\frac{n^3}{(n+2)(n+5)(n+7)}\right]^{1/3} = 1^{1/3} = 1 < \infty$$

And $\sum \frac{1}{n}$ dVG (p-series, p=1) so $\sum u_n$ dvg.

Series B is convergent by AST since $u_n \xrightarrow{n\to\infty} 0$.

15. See 2013S, Fall Exam, #21.

16. If we let $u_n = (-1)^n(x-3)^n/(n+1)$, then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+2} \cdot \frac{n+1}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n+1}{n+2}(x-3)\right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)|x-3|$$

$$= (1)|x-3| = |x-3|.$$

16 - continued

By the Ratio Test the series is absolutely convergent if $|x-3|<1$, that is, if

$$-1 < x-3 < 1 \quad \text{or} \quad 2 < x < 4.$$

The series diverges if $x<2$ or $x>4$. The numbers 2 and 4 must be checked separately. If $x=4$ the resulting series is

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} + \cdots$$

which converges by the Alternating Series Test. For $x=2$ the series becomes

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + \cdots$$

which is the divergent harmonic series. Hence the interval of convergence is $(2, 4]$.

17. From textbook, § 11.9, # 19

$$f(x) = \frac{x}{x^2+16} = \frac{x}{16}\left(\frac{1}{1-(-x^2/16)}\right) = \frac{x}{16}\sum_{n=0}^{\infty}\left(-\frac{x^2}{16}\right)^n = \frac{x}{16}\sum_{n=0}^{\infty}(-1)^n \frac{1}{16^n}x^{2n} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{16^{n+1}}x^{2n+1}.$$

Geometric series, valid for $r = \frac{-x^2}{16} \in (-1,1)$.

The series converges when

$$|r| = |-x^2/16| < 1 \quad \Leftrightarrow \quad x^2 < 16 \quad \Leftrightarrow \quad |x| < 4,$$

18. From textbook, § 11.9, Example 8.

The first step is to express the integrand, $1/(1+x^7)$, as the sum of a power series. For this we use the geometric series with ratio r & $\boxed{\text{replace } r \text{ by } -x^7.}$

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty}(-x^7)^n$$

valid when $|r| = |-x^7| < 1$

$$= \sum_{n=0}^{\infty}(-1)^n x^{7n} = 1 - x^7 + x^{14} - \cdots$$

Now we integrate term by term:

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{7n+1}}{7n+1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$. , i.e. $x \in (-1, 1)$. Note $0 \in (-1,1)$.

So for $t \in (-1,1)$,

$$f(t) = \int_0^t \frac{dx}{1+x^7} = \sum_{n=0}^{\infty}\left((-1)^n \frac{x^{7n+1}}{7n+1}\right)\Bigg|_0^t = \sum_{n=0}^{\infty}\frac{(-1)^n t^{7n+1}}{7n+1} - 0.$$

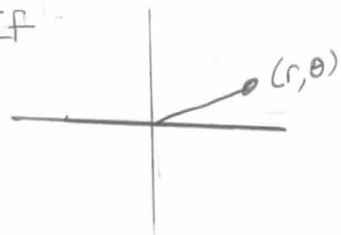
19. See 2014S, Exam 3, # 3.

20 See 2012F, Practice Exam 3, # 3

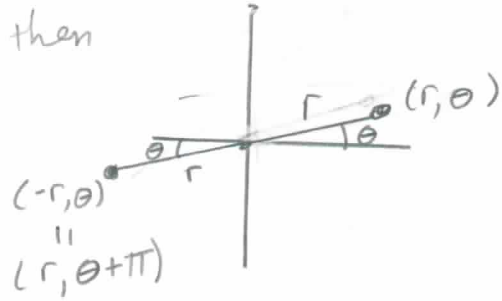
21. " , # 4

22. " , # 5.

23. If



, then



so (r,θ) can be represented as $(-r, \theta+\pi)$.

So can see geometrically. Algebraically, let (r,θ) have cartesian Coordinates (x, y) and $(-r, \theta+\pi)$ " (x_0, y_0). Then

$$x_0 = -r\cos(\theta+\pi) = -r(-\cos\theta) = r\cos\theta = x,$$

$$y_0 = -r\sin(\theta+\pi) = -r(-\sin\theta) = r\sin\theta = y.$$

So $(x,y) = (x_0, y_0)$.

24. See 2011F, Final Exam, # 24

25. See 2011F, Final Exam, # 25.