1. Suppose that the power series $\sum_{n=1}^{\infty} c_n(x-10)^n$ has interval of convergence $(1, 19)$ What is the interval of convergence of the power series $\sum_{n=1}^{\infty} c_n x^{2n}$?

a. $[9, 19]$ b. $[-3, 3]$ c. $(-3, 3)$ d. $(-81, 81)$ e. none of the others.

$\sum c_n x^{2n}$ converges $\Leftrightarrow \sum c_n (x^2)^n$ conv. $\Leftrightarrow \sum c_n [(x^2+10)-10]^n$ conv.

$\xLeftrightarrow{\text{given}}$ $+1 < x^2 + 10 < 19 \Leftrightarrow -9 < x^2 < 9 \Leftrightarrow -3 < x < 3.$

2. Find a power series representation for $f(x) = \ln(10-x)$. Hint: $\ln(ab) = \ln a + \ln b$.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n 10^n}$ b. $\sum_{n=1}^{\infty} \frac{10x^n}{n^n}$ c. $\ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{10^n}$ d. $\ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{n\,10^n}$ e. none of the others.

$\ln(10-x) = \ln\left(10\left(1-\frac{x}{10}\right)\right) = \ln 10 + \ln\left[1+\left(\frac{-x}{10}\right)\right]$ ← used a "commonly used Taylor Series"

$= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{-x}{10}\right)^n = \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} \frac{x^n}{10^n}$

$= \ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{n\,10^n}$

$(-1)^{n+1}(-1)^n = (-1)^{2n+1} = (-1)^{2n}(-1)^1 = -1.$

I forgot to ask when is this expansion valid so let's do:

Valid $\Leftrightarrow -1 < \frac{-x}{10} \leq 1 \Leftrightarrow -10 \leq x < 10$

3. Suppose that the interval of convergence of the series $\sum_{n=1}^{\infty} c_n(x-x_0)^n$ is $(x_0 - R, x_0 + R]$. What can be said about the series at $x_0 + R$?

a. It must be absolutely convergent. b. It must be conditionally convergent.

c. It must be divergent. d. Nothing can be said. e. none of the others.

This was taken from our homework for §11.8. Solution is posted on our homework page.

Problem source is Anton, et. al., 8th ed., §10.8 # 63.

63. Prove: If the interval of convergence of the series $\sum_{k=0}^{\infty} c_k(x-x_0)^k$ is $(x_0 - R, x_0 + R]$, then the series converges conditionally at $x_0 + R$.

63. The assumption is that $\sum_{k=0}^{\infty} c_k R^k$ is convergent and $\sum_{k=0}^{\infty} c_k(-R)^k$ is divergent. Suppose that $\sum_{k=0}^{\infty} c_k R^k$ is absolutely convergent then $\sum_{k=0}^{\infty} c_k(-R)^k$ is also absolutely convergent and hence convergent because $|c_k R^k| = |c_k(-R)^k|$, which contradicts the assumption that $\sum_{k=0}^{\infty} c_k(-R)^k$ is divergent so $\sum_{k=0}^{\infty} c_k R^k$ must be conditionally convergent.

4. Find the 4^{th} order Maclaurin polynomial for $f(x) = \frac{1}{(x+1)^3}$.

a. $-3x+6x^2-10x^3+15x^4$ b. $1-3x+6x^2-10x^3+15x^4$ c. $1-3(x+1)+6(x+1)^2-10(x+1)^3+15(x+1)^4$

d. $1-3x+12x^2-30x^3+360x^4$ e. none of the others.

Way #1.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$C_n = \frac{f^{(n)}(0)}{n!}$
0	$(x+1)^{-3}$	1	$\frac{1}{0!} = 1$
1	$-3(x+1)^{-4}$	-3	$\frac{-3}{1!} = -3$
2	$+3\cdot4(x+1)^{-5}$	$+3\cdot4$	$\frac{3\cdot4}{2!} = \frac{3\cdot\cancel{4}^{2}}{\cancel{2}} = 6$
3	$-3\cdot4\cdot5(x+1)^{-6}$	$-3\cdot4\cdot5$	$\frac{-3\cdot4\cdot5}{3!} = \frac{-\cancel{3}\cdot\cancel{4}^{2}\cdot5}{1\cdot\cancel{2}\cdot\cancel{3}} = -10$
4	$+3\cdot4\cdot5\cdot6(x+1)^{-7}$	$+3\cdot4\cdot5\cdot6$	$\frac{3\cdot4\cdot5\cdot6}{4!} = \frac{\cancel{3}\cdot\cancel{4}\cdot5\cdot\cancel{6}^{3}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}} = 15$

Note: Use the term Maclaurin so we know the center is $x_0 = 0$.

So the answer is b.

5. Consider the function $f(x) = e^x$ over the interval $(-1, 3)$. The 4^{th} order Taylor polynomial of $y = f(x)$ about the center $\boxed{x_0 = 0}$ is

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = \sum_{n=0}^{4} \frac{x^n}{n!}.$$

The 4^{th} order Remainder term $R_4(x)$ is defined by $R_4(x) = f(x) - P_4(x)$ and so $e^x \approx P_4(x)$ where the approximation is within an error of $|R_4(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_4(x)|$ that is valid for each $x \in (-1, 3)$.

a. $\frac{(e^3)(3^4)}{4!}$ b. $\frac{(e^{-1})(3^4)}{4!}$ c. $\frac{(e^3)(3^5)}{5!}$ d. $\frac{(e^{-1})(3^5)}{5!}$ e. none of the others.

For each $x \in (-1, 3)$, the exists c between x & x_0 (so c is also in $(-1, 3)$) so that:

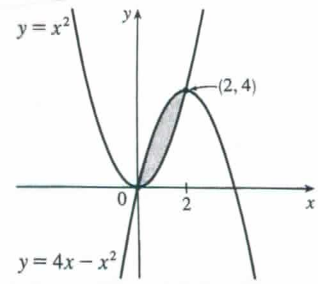
$$|R_4(x)| \underset{\text{theorm}}{\overset{\text{big}}{=}} \left|\frac{f^{(5)}(c)}{5!} x^5\right| = \frac{e^c |x|^5}{5!} \leq \frac{e^3 \cdot 3^5}{5!}$$

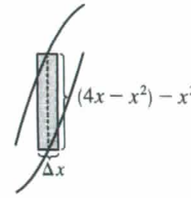
6. Express the area of the region enclosed by $y = x^2$ and $y = 4x - x^2$ as an integral.

a. $\int_0^4 \left[(4x - x^2) - x^2\right]\,dx$ b. $\int_0^4 \left[x^2 - (4x - x^2)\right]\,dx$ c. $\int_0^2 \left[(4x - x^2) - x^2\right]\,dx$
d. $\int_0^2 \left[x^2 - (4x - x^2)\right]\,dx$ e. none of the others.

Problem Source: § 6.1 Exercise # 12.

12. $x^2 = 4x - x^2 \;\Leftrightarrow\; 2x^2 - 4x = 0 \;\Leftrightarrow\; 2x(x-2) = 0 \;\Leftrightarrow\; x = 0$ or 2, so





7. Let R be the region bounded by the curves

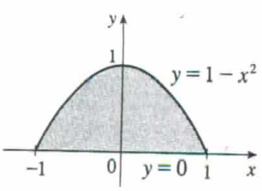
$$y = 1 - x^2 \quad \text{and} \quad y = 0\,.$$

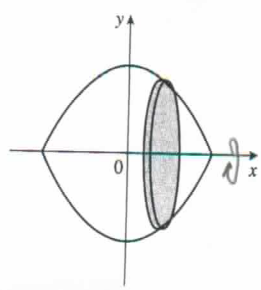
Express as an integral the volume of the solid generated by revolving R about the x-axis.

a. $\pi \int_0^1 \sqrt{1-y}\,dy$ b. $2\pi \int_0^1 y\sqrt{1-y}\,dy$ c. $\pi \int_0^1 (1-x^2)^2\,dx$ d. $\pi \int_{-1}^1 (1-x^2)^2\,dx$
e. none of the others.

Problem Source: § 6.2 Exercise # 2.

Volume of typical disk $= \pi r^2 h$
$= \pi (1-x^2)^2 dx$





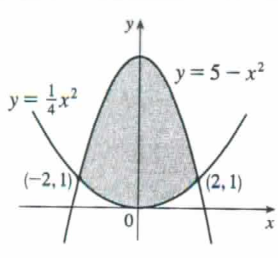
8. Let R be the region bounded by the curves

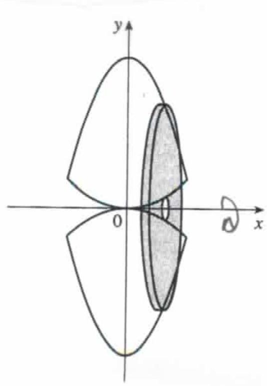
$$y = \frac{x^2}{4} \quad \text{and} \quad y = 5 - x^2\,.$$

Express as integral(s) the volume of the solid generated by revolving R about the x-axis.

a. $\pi \int_{-2}^2 \left[(5 - x^2) - \left(\frac{x^2}{4}\right)\right]^2 dx$ b. $\pi \int_{-2}^2 \left[(5 - x^2)^2 - \left(\frac{x^2}{4}\right)^2\right] dx$ c. $2\pi \int_0^5 2y\sqrt{5-y}\,dy$
d. $2\pi \int_0^1 y\sqrt{4y}\,dy \;+\; 2\pi \int_0^1 y\sqrt{5-y}\,dy$ e. none of the others.

Problem Source: § 6.2 Exercise # 8.





Volume of typical washer
$= \pi \left[(\text{radius}_{\text{big}}^2 - \text{radius}_{\text{little}}^2) \right] \text{height}$
$= \pi \left[(5 - x^2)^2 - (\tfrac{1}{4}x^2)^2 \right] \Delta x$

9. Let R be the region bounded by the curves

$$y = \frac{1}{x} \quad \text{and} \quad y = 0 \quad \text{and} \quad x = 1 \quad \text{and} \quad x = 2\,.$$

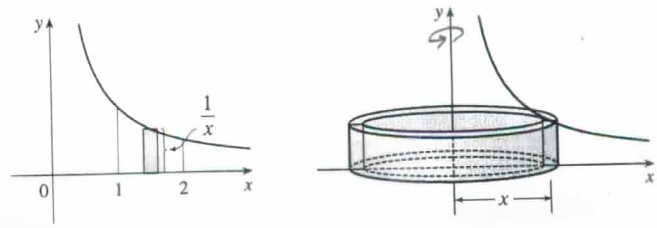
Express as integral(s) the volume of the solid generated by revolving R about the y-axis.

a. $2\pi \int_1^2 \frac{1}{x}\,dx$ b. $2\pi \int_1^2 x\,dx$ c. $2\pi \int_1^2 1\,dx$ d. $\pi \int_0^1 \left[\left(\frac{1}{y}\right)^2 - (1)^2\right] dy$

e. none of the others.

Problem Source: § 6.3 Exercise # 3.

3. $V = \int_1^2 2\pi x \cdot \frac{1}{x}\,dx = 2\pi \int_1^2 1\,dx$



Volume of typical shell $= 2\pi\,\underbrace{(\text{avg. radius})}_{x}\,\underbrace{(\text{height})}_{\frac{1}{x}}\,\underbrace{(\text{thickness})}_{\Delta x}$

$= 2\pi \quad x. \quad \frac{1}{x} \quad \Delta x \quad = 2\pi\,\Delta x.$

10. Let R be the region bounded by the curves

$$y = x \quad \text{and} \quad y = 4x - x^2\,.$$

Express as integral(s) the volume of the solid generated by revolving R about the line $x = 7$.

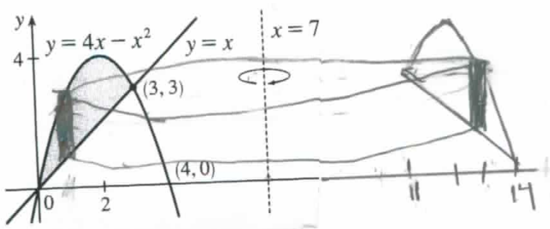
a. $2\pi \int_0^3 x\left[x - (4x - x^2)\right] dx$ b. $2\pi \int_0^3 x\left[(4x - x^2) - x\right] dx$

c. $2\pi \int_0^3 (7 - x)\left[x - (4x - x^2)\right] dx$ d. $2\pi \int_0^3 (7 - x)\left[(4x - x^2) - x\right] dx$

e. none of the others.

Problem Source: § 6.3 Exercise # 22.

22. $V = \int_0^3 2\pi(7 - x)[(4x - x^2) - x]\,dx$



Volume of typical shell $= 2\pi\,\underbrace{(\text{avg. radius})}_{(7-x)}\,\underbrace{(\text{height})}_{[(4x-x^2)-x]}\,\underbrace{(\text{thickness})}_{(\Delta x)}$

$= 2\pi \;(7-x)\;\left[(4x-x^2)-x\right](\Delta x)$