HAND IN PART

Prof. Girardi	Math 142	Spring 2014	04.17.2014	Exam 3

MARK BOX		
PROBLEM	POINTS POSSIBLE	YOUR SCORE
0A	9	
0B	8	
0C	10	
0D	12	
Total for 0	39	
Total for 1–10	61	
%	100	

NAME: solution key

PIN: ______________________

INSTRUCTIONS

(1) The MARK BOX above indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(2) You may **not** use an electronic device, a calculator, books, personal notes.
(3) As you know, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.
(4) Hand in the HAND IN PART. You can take the STATEMENT OF MULTIPLE CHOICE PROBLEMS part home with you (so you can check your answers once the solutions are posted).
(5) During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.
(6) This exam covers (from *Calculus* by Stewart, 6^{th} ed., ET): §11.8 – 11.11, 6.1 – 6.3 .

0A. Power Series Condsider a (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \,(x - x_0)^n \;, \tag{1}$$

with radius of convergence $R \in [0, \infty]$. (Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

- For the next 4 questions, circle one and only one choice.
  Abbreviations: AC for absolutely convergent, CC for conditionally convergent, and DV for divergent.
  (1) For $x = x_0$, the power series $h(x)$ in (1)
  ⓐ is always AC b. is always CC c. is always DV
  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.
  (2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series $h(x)$ in (1)
  ⓐ is always AC b. is always CC c. is always DV
  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.
  (3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$ the power series $h(x)$ in (1)
  a. is always AC b. is always CC ⓒ is always DV
  d. can do anything, i.e., there are examples showing that it can be AC, CC, or DV.
  (4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series $h(x)$ in (1)
  a. is always AC b. is always CC c. is always DV
  ⓓ can do anything, i.e., there are examples showing that it can be AC, CC, or DV.

- For the next 2 problems, let $R > 0$ and fill-in the boxes. Consider the function $y = h(x)$ defined by the power series in (1).
  (1) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

  $$h'(x) \;=\; \sum_{n=0}^{\infty} \boxed{n\, a_n \,(x - x_0)^{n-1}} \;. \tag{2}$$

  What can you say about the radius of convergence of the power series in (2)? $\boxed{\text{It's the same } R}$.
  (2) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

  $$\int_{x=\alpha}^{x=\beta} h(x)\, dx \;=\; \sum_{n=0}^{\infty} \boxed{\frac{a_n}{n+1}\;(x - x_0)^{n+1}}\;\Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

0B. Taylor/Maclaurin Polynomials and Series. Fill-in the boxes.

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. The formula for c_n is

$$c_n = \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

b. In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) = \boxed{f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N}$$

c. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_N(x) = \boxed{\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n}$$

d. In open form (i.e., with ... and without a $\sum$-sign)

$$P_\infty(x) = \boxed{f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots}$$

e. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n}$$

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!} (x - x_0)^{(N+1)}} \text{ for some } c \text{ between } \boxed{x} \text{ and } \boxed{x_0}.$$

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.

0C. Area and Volume of Revolutions. Let's start with some region R in the (2 dimensional) xy-plane and revolve R around an axis of revolution to generate a (3 dimensional) solid of revolution S. Next we want to find the area of R as well as the volume of S.

- **In parts a, fill in the boxes with:** x **or** y**.**
- **In parts b, c, and d, fill in the boxes with a formula involving *some* of:** **2 , π , radius , base , radius$_{\text{big}}$, radius$_{\text{little}}$, average radius , height , and/or thickness .**

►. **Area via Riemann Sums**. Let's find the area of R by forming typical rectangles.

a. We first partition either the [x]-axis or the [y]-axis. (We can pick either.)

⊡. Next, using the partition, we form typical rectangles. Then we find the area of each typical rectangle.

b. If we partition the z-axis, where z is either x or y, the $\Delta z =$ [base] of a typical rectangle.

c. The area of a typical rectangle is [(height) (base)].

►. **Disk/Washer Method**. Let's find the volume of the solid of revolution S using the disk/washer method.

a. If the axis of revolution is:

- the x-axis, or parallel to the x-axis, then we partition the [x]-axis.
- the y-axis, or parallel to the y-axis, then we partition the [y]-axis.

⊡. Next, using the partition, we form typical disk/washer's. Then we find the volume of each typical disk/washer.

b. If we partition the z-axis, where z is either x or y, the $\Delta z =$ [height] of a tyical disk/washer.

c. If we use the **disk method**, then the volume of a typical disk is:

$$\pi\ (\text{radius})^2\ (\text{height})$$.

d. If we use the **washer method**, then the volume of a typical washer is: (either form is fine)

$$\pi\ (\text{radius}_{\text{big}})^2\ (\text{height})\ -\ \pi\ (\text{radius}_{\text{little}})^2\ (\text{height})\ \overset{\text{or}}{=}\ \pi\ \left[(\text{radius}_{\text{big}})^2 - (\text{radius}_{\text{little}})^2\right]\ (\text{height})$$.

►. **Shell Method**. Let's find the volume of this solid of revolution S using the shell method.

a. If the axis of revolution is:

- the x-axis, or parallel to the x-axis, then we partition the [y]-axis.
- the y-axis, or parallel to the y-axis, then we partition the [x]-axis.

⊡. Next, using the partition, we form typical shells. Then we find the volume of each typical shell.

b. If we partition the z-axis, where z is either x or y, the $\Delta z =$ [thickness] of a typical shell.
Also acceptable is $\Delta z = \text{radius}_{\text{big}} - \text{radius}_{\text{little}}$.

c. The volume of a typical shell is: (either form is fine)

$$2\pi\ (\text{average radius})\ (\text{height})\ (\text{thickness})\ \overset{\text{or}}{=}\ 2\pi\ (\text{radius})\ (\text{height})\ (\text{thickness})$$

.

0D. Commonly Used Taylor Series Fill in the 12 below blanks boxes with the choices a – ℓ. You may use a choice more then once or not at all.

The 12 blank boxes.

- A power series expansion for $y = \cos x$ is [d] and is valid precisely when [g].
- A power series expansion for $y = \sin x$ is [e] and is valid precisely when [g].
- A power series expansion for $y = e^x$ is [b] and is valid precisely when [g].
- A power series expansion for $y = \frac{1}{1-x}$ is [a] and is valid precisely when [h].
- A power series expansion for $y = \ln(1+x)$ is [c] and is valid precisely when [j].
- A power series expansion for $y = \tan^{-1} x$ is [f] and is valid precisely when [i].

The choices a-l for above 12 blank boxes.

a. $\sum_{n=0}^{\infty} x^n$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

c. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

e. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

f. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

g. $x \in \mathbb{R}$

h. $(-1, 1)$

i. $[-1, 1]$

j. $(-1, 1]$

k. $[-1, 1)$

ℓ. none of the others

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS

Instructions.

- Indicate (by circling) directly in the table below your solution to each problem.
- You may choice up to 3 answers for each problem. The scoring is as follows.
  - For a problem with precisely one answer marked and the answer is correct, 6 points.
  - For a problem with precisely two answers marked, one of which is correct, 3 points.
  - For a problem with precisely three answers marked, one of which is correct, 1 point.
  - All other cases, 0 points.
- Turn in this Hand In Part of the test.
- As for the Statement of Mulitple Choice Problem of the exam, note the following.
  - Do NOT hand it in. Take it home with you.
  - Use the back blank sides for scratch paper.
- Fill in the "number of solutions circled" column. (Worth 1 point)

Your Solutions							
PROBLEM						# of solutions circled	points
1	1a	1b	(1c)	1d	1e		
2	2a	2b	2c	(2d)	2e		
3	3a	(3b)	3c	3d	3e		
4	4a	(4b)	4c	4d	4e		
5	5a	5b	(5c)	5d	5e		
6	6a	6b	(6c)	6d	6e		
7	7a	7b	7c	(7d)	7e		
8	8a	(8b)	8c	8d	8e		
9	9a	9b	(9c)	9d	9e		
10	10a	10b	10c	(10d)	10e		
TOTAL POINT							

STATEMENT OF MULTIPLE CHOICE PROBLEMS

1. Suppose that the power series $\sum_{n=1}^{\infty} c_n(x-10)^n$ has interval of convergence $(1, 19)$ What is the interval of convergence of the power series $\sum_{n=1}^{\infty} c_n x^{2n}$?

a. $[9, 19]$ b. $[-3, 3]$ c. $(-3, 3)$ d. $(-81, 81)$ e. none of the others.

2. Find a power series representation for $f(x) = \ln(10 - x)$. Hint: $\ln(ab) = \ln a + \ln b$.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n10^n}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{10x^n}{n^n}$ c. $\displaystyle\ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{10^n}$ d. $\displaystyle\ln 10 - \sum_{n=1}^{\infty} \frac{x^n}{n\,10^n}$ e. none of the others.

3. Suppose that the interval of convergence of the series $\sum_{n=1}^{\infty} c_n(x-x_0)^n$ is $(x_0 - R, x_0 + R]$. What can be said about the series at $x_0 + R$?

a. It must be absolutely convergent. b. It must be conditionally convergent.

c. It must be divergent. d. Nothing can be said. e. none of the others.

This was taken from our homework for §11.8. Solution is posted on our homework page.

Problem source is Anton, et. al., 8^{th} ed., §10.8 # 63.

4. Find the 4^{th} order Maclaurin polynonial for $f(x) = \frac{1}{(x+1)^3}$.

a. $-3x+6x^2-10x^3+15x^4$ b. $1-3x+6x^2-10x^3+15x^4$ c. $1-3(x+1)+6(x+1)^2-10(x+1)^3+15(x+1)^4$

d. $1 - 3x + 12x^2 - 30x^3 + 360x^4$ e. none of the others.

5. Consider the function $f(x) = e^x$ over the interval $(-1, 3)$. The 4^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = \sum_{n=0}^{4} \frac{x^n}{n!} .$$

The 4^{th} order Remainder term $R_4(x)$ is defined by $R_4(x) = f(x) - P_4(x)$ and so $e^x \approx P_4(x)$ where the approximation is within an error of $|R_4(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_4(x)|$ that is valid for each $x \in (-1, 3)$.

a. $\frac{(e^3)(3^4)}{4!}$ b. $\frac{(e^{-1})(3^4)}{4!}$ c. $\frac{(e^3)(3^5)}{5!}$ d. $\frac{(e^{-1})(3^5)}{5!}$ e. none of the others.

More Problems On Next Page

6. Express the area of the region enclosed by $y = x^2$ and $y = 4x - x^2$ as an integral.

a. $\int_0^4 \left[\left(4x - x^2\right) - x^2\right] dx$ b. $\int_0^4 \left[x^2 - \left(4x - x^2\right)\right] dx$ c. $\int_0^2 \left[\left(4x - x^2\right) - x^2\right] dx$
d. $\int_0^2 \left[x^2 - \left(4x - x^2\right)\right] dx$ e. none of the others.

Problem Source: § 6.1 Exercise # 12.

7. Let R be the region bounded by the curves

$$y = 1 - x^2 \qquad \text{and} \qquad y = 0\ .$$

Express as an integral the volume of the solid generated by revolving R about the x-axis.

a. $\pi \int_0^1 \sqrt{1-y}\; dy$ b. $2\pi \int_0^1 y\sqrt{1-y}\; dy$ c. $\pi \int_0^1 (1-x^2)^2\, dx$ d. $\pi \int_{-1}^1 (1-x^2)^2\, dx$
e. none of the others.

Problem Source: § 6.2 Exercise # 2.

8. Let R be the region bounded by the curves

$$y = \frac{x^2}{4} \qquad \text{and} \qquad y = 5 - x^2\ .$$

Express as integral(s) the volume of the solid generated by revolving R about the x-axis.

a. $\pi \int_{-2}^2 \left[\left(5 - x^2\right) - \left(\frac{x^2}{4}\right)\right]^2 dx$ b. $\pi \int_{-2}^2 \left[\left(5 - x^2\right)^2 - \left(\frac{x^2}{4}\right)^2\right] dx$ c. $2\pi \int_0^5 2y\sqrt{5-y}\; dy$
d. $2\pi \int_0^1 y\sqrt{4y}\; dy \;+\; 2\pi \int_0^1 y\sqrt{5-y}\; dy$ e. none of the others.

Problem Source: § 6.2 Exercise # 8.

9. Let R be the region bounded by the curves

$$y = \frac{1}{x} \qquad \text{and} \qquad y = 0 \qquad \text{and} \qquad x = 1 \qquad \text{and} \qquad x = 2\ .$$

Express as integral(s) the volume of the solid generated by revolving R about the y-axis.

a. $2\pi \int_1^2 \frac{1}{x}\, dx$ b. $2\pi \int_1^2 x\, dx$ c. $2\pi \int_1^2 1\, dx$ d. $\pi \int_0^1 \left[\left(\frac{1}{y}\right)^2 - (1)^2\right] dy$
e. none of the others.

Problem Source: § 6.3 Exercise # 3.

10. Let R be the region bounded by the curves

$$y = x \qquad \text{and} \qquad y = 4x - x^2\ .$$

Express as integral(s) the volume of the solid generated by revolving R about the line $x = 7$.

a. $2\pi \int_0^3 x \left[x - \left(4x - x^2\right)\right] dx$ b. $2\pi \int_0^3 x \left[\left(4x - x^2\right) - x\right] dx$
c. $2\pi \int_0^3 (7 - x) \left[x - \left(4x - x^2\right)\right] dx$ d. $2\pi \int_0^3 (7 - x) \left[\left(4x - x^2\right) - x\right] dx$
e. none of the others.

Problem Source: § 6.3 Exercise # 22.