1. $$\int_0^1 x^3 (x^4+1)^2\, dx = \frac{1}{4}\int_{x=0}^{x=1} (x^4+1)^2 (4x^3\, dx) =$$

$$\boxed{\begin{array}{l} u = x^4+1 \\ du = 4x^3 \end{array}}$$

$$= \frac{1}{4}\int_{u=1}^{u=2} u^2\, du = \frac{1}{4}\, \frac{u^3}{3} \Big|_{u=1}^{u=2}$$

$$= \frac{1}{12}\left[2^3 - 1\right] = \boxed{\frac{7}{12}}$$

2. From 100 integrals # 23.

[redacted] ~~23~~ $$\int \ln(1+x)\, dx = x\ln(1+x) - \int \frac{x}{1+x}\, dx \overset{LD}{=} x\ln(1+x) - \int \frac{1+x-1}{1+x}\, dx$$

$$\boxed{\begin{array}{ll} u = \ln(1+x) & dv = dx \\ du = \frac{dx}{1+x} & v = x \end{array}}$$

$$= x\ln(1+x) - \int \left(1 - \frac{1}{1+x}\right) dx$$

$$= x\ln(1+x) - x + \ln(1+x) + C$$

$$= \boxed{(x+1)\ln(1+x) - x + C}$$

---

or

$$u = \ln(1+x) \qquad dv = dx$$

$$du = \frac{1}{1+x}\, dx \qquad v = 1+x$$

$$\int \ln(1+x)\, dx = (1+x)\ln(1+x) - \int \frac{1+x}{1+x}\, dx$$

$$= (1+x)\ln(1+x) - \int dx$$

$$= \boxed{(1+x)\ln(1+x) - x + C}$$

So $$\int_0^1 \ln(1+x)\, dx = \left[(1+x)\ln(1+x) - x\right]\Big|_0^1$$

$$= \left[2\ln 2 - 1\right] - \left[1\underbrace{\ln 1}_{=0} - 0\right] = \boxed{2\ln 2 - 1}$$

3. Example from book, page 453-4.

Let

$$u = x \qquad dv = \sin x\, dx$$

Then

$$du = dx \qquad v = -\cos x$$

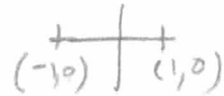
and so

$$\int x \sin x\, dx = \int \overbrace{x}^{u}\ \overbrace{\sin x\, dx}^{dv} = \overbrace{x}^{u}\ \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\ \overbrace{dx}^{du}$$

$$= -x\cos x + \int \cos x\, dx$$

$$= -x\cos x + \sin x + C$$

So $\int_0^{\pi} x\sin x\,dx = (\sin x - x\cos x)\Big|_{x=0}^{x=\pi}$



$= [0-\pi(-1)] - [0-0(1)] = \pi.$

4. $\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x\,(1-\sin^2 x)\,\boxed{\cos x\,dx}$

$\boxed{\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array}}$

$= \int t^2(1-t^2)\,dt = \int (t^2 - t^4)\,dt$

$= \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$

Check $D_x\left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right] = \frac{1}{3}\cdot 3\cdot \sin^2 x\cos x - \frac{1}{5}\,5\sin^4 x\cos x$

$= (\sin^2 x)(\cos x)[1-\sin^2 x] = \sin^2 x\cos^3 x.$ ✓

So $\int_0^{\pi/2} \sin^2 x\cos^3 x\,dx = \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]\Big|_{x=0}^{x=\pi/2}$

$= \left[\frac{1}{3} - \frac{1}{5}\right] - \left[\frac{0}{3} - \frac{0}{5}\right] = \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15} = \frac{2}{15}.$

5. From book, p464-5

EXAMPLE ~~8~~ Find $\int \sec^3 x\,dx$.

SOLUTION Here we integrate by parts with

$$u = \sec x \qquad dv = \sec^2 x\,dx$$
$$du = \sec x\tan x\,dx \qquad v = \tan x$$

Then
$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\tan x - \int \sec x\tan^2 x\,dx \\ &= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx \\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx\end{aligned}$$

Using Formula 1 and solving for the required integral, we get

$$\int \sec^3 x\,dx = \tfrac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + C$$

So $\int_0^{\pi/4} \sec^3 x\,dx = \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|)\Big|_{x=0}^{x=\pi/4}$

$= \frac{1}{2}\left((\sqrt{2})(1) + \ln|\sqrt{2}+1|\right) - \frac{1}{2}\left((1)(0) + \underbrace{\ln|1+0|}_{0}\right)$

$= \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right).$

6. From book, p469

Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$

SOLUTION Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

Thus we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2} \\ &= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C \\ &= -\frac{\csc\theta}{4} + C\end{aligned}$$

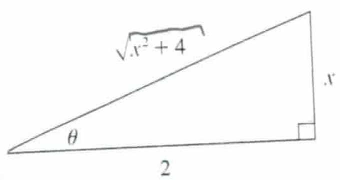


FIGURE 3
$\tan\theta = \frac{x}{2}$

We use Figure 3 to determine that $\csc\theta = \sqrt{x^2+4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C$$

So $\displaystyle\int_1^3 \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x}\Bigg|_{x=1}^{x=3}$

$\displaystyle= \frac{\sqrt{x^2+4}}{4x}\Bigg|_{x=3}^{x=1} = \frac{\sqrt{5}}{4} - \frac{\sqrt{13}}{12}.$

7 A homework problem : §7.4 # 11.

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1 = A(x-1) + B(x+1)$. Substituting 1 for x gives $1 = 2B \Leftrightarrow B = \frac{1}{2}$. Substituting -1 for x gives $1 = -2A \Leftrightarrow A = -\frac{1}{2}$. Thus,

$$\begin{aligned}\int_2^3 \frac{1}{x^2-1}\,dx &= \int_2^3 \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1|\right]_2^3 \\ &= \left(-\tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2\right) - \left(-\tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2 + \ln 3 - \ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]\end{aligned}$$

8.)

Problem Source: Textbook, § 7.4, # 15

$$\boxed{\int \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = x + \ln|x| - \frac{2}{x} - \ln|x-2| + C}$$

PFD

Hint: Do we have (Strictly) Bigger Bottoms ?

NO so need to do long division ... but it's easy to "fake" long division on this one (lucky us)

$$\frac{x^3-2x^2-4}{x^3-2x^2} = \frac{x^3-2x^2}{x^3-2x^2} + \frac{-4}{x^3-2x^2} = 1 + \frac{-4}{x^3-2x^2}$$

Find PFD for $\frac{-4}{x^3-2x^2} = \frac{-4}{x^2(x-2)} = \frac{-4}{(x-0)^2(x-2)}$

(linear term)^2 (linear term)^1

$$\frac{-4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$$

$$\Rightarrow \boxed{-4 = Ax(x-2) + B(x-2) + Cx^2}$$

$x=0$: $-4 = -2B \Rightarrow \boxed{B=2}$

$x=2$: $-4 = C\cdot 2^2 \Rightarrow \boxed{C=-1}$

equate coeff

x^2: $0 = A + C \xrightarrow{C=-1} \boxed{A = 1}$

x^1: $0 = -2A + B$

x^0: $-4 = -2B$

$$\int \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int \left[1 + \frac{1}{x} + \frac{2}{x^2} + \frac{-1}{x-2}\right] dx$$

$$\Downarrow$$

$$\int 2x^{-2}\,dx = \frac{2x^{-1}}{-1} + C$$

So

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \left[4 + \ln 4 - \frac{2}{4} - \ln 2\right] - \left[3 + \ln 3 - \frac{2}{3} - \ln 1\right]$$

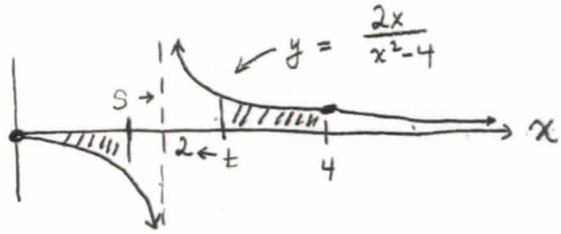
$$= \ln 4 - \ln 2 - \ln 3 + 4 - \frac{1}{2} - 3 + \frac{2}{3} = \left(\ln\frac{4}{6}\right) + \frac{7}{6}$$

/9

Math 142, Spring 2014, Sol'n to Exam I p 5/6

#9 Was an example from class. 7.32

To help with Example 8, let's first make a rough sketch of the graph of $f(x) = \frac{2x}{x^2-4}$ for $x \geq 0$.
The domain of $y = f(x)$ is $[0, \infty) \setminus \{2\} = [0, 2) \cup (2, \infty)$.
$\lim_{x \to 2^+} \frac{2x}{x^2-4} = ?$ and $\lim_{x \to 2^-} \frac{2x}{x^2-4} = ?$
Next you can do the 1st Derivative Test to see when f is increasing and decreasing.
Then you can do the 2st Derivative Test to see when f is CCU and CCD.
If you need to review your Calc I for graphing - do it!



Ex 8. $\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4} = ?$

$$\begin{aligned}
\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4} &= \left[\lim_{s \to 2^-} \int_{x=0}^{x=s} \frac{2x\,dx}{x^2-4}\right] + \left[\lim_{t \to 2^+} \int_{x=t}^{x=4} \frac{2x\,dx}{x^2-4}\right] \\
&= \left[\lim_{s \to 2^-} \ln|x^2-4| \,\Big|_{x=0}^{x=s}\right] + \left[\lim_{t \to 2^-} \ln|x^2-4| \,\Big|_{x=t}^{x=4}\right] \\
&= \underbrace{\left[\lim_{s \to 2^-} \left(\ln|s^2-4| - \ln 4\right)\right]}_{s \to 2^- \Rightarrow |s^2-4| \to 0^+ \Rightarrow \ln|s^2-4| \to -\infty} + \underbrace{\left[\lim_{t \to 2^-} \left(\ln 12 - \ln|t^2-4|\right)\right]}_{t \to 2^+ \Rightarrow |t^2-4| \to 0^+ \Rightarrow \ln|t^2-4| \to -\infty} \\
&= [-\infty] + [+\infty]\,. \quad \Leftarrow \text{THIS DOES NOT MAKE SENSE !}
\end{aligned}$$

So $\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4}$ diverges (or can also say DNE).

Ex 8. Revisited What is wrong with this way?

$$\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4} = \ln|x^2-4| \,\Big|_{x=0}^{x=4} = \ln 12 - \ln 4 = \ln \frac{12}{4} = \ln 3\,.$$

A common mistake is to NOT recognize an improper integral when you see him and then just (incorrectly) *blindly* integrate.

Ex 8. ReRevisited

$$\begin{aligned}
\int_{x=0}^{x=\infty} \frac{2x\,dx}{x^2-4} &= \int_{x=0}^{x=2} \frac{2x\,dx}{x^2-4} + \int_{x=2}^{x=17} \frac{2x\,dx}{x^2-4} + \int_{x=17}^{x=\infty} \frac{2x\,dx}{x^2-4} \\
&= \underbrace{\left[\lim_{s \to 2^-} \int_{x=0}^{x=s} \frac{2x\,dx}{x^2-4}\right]}_{-\infty} + \underbrace{\left[\lim_{t \to 2^+} \int_{x=t}^{x=17} \frac{2x\,dx}{x^2-4}\right]}_{\infty} + \underbrace{\left[\lim_{u \to \infty} \int_{x=17}^{x=u} \frac{2x\,dx}{x^2-4}\right]}_{\text{who cares}}
\end{aligned}$$

So $\int_{x=0}^{x=\infty} \frac{2x\,dx}{x^2-4}$ diverges (or can also say DNE).

 # 10 $\left(1+\frac{c}{n}\right)^n$ < $\xrightarrow{\text{as } n\to\infty} 1^\infty$, ind. form, so... >

thinking land

Let $y = \left(1+\frac{c}{x}\right)^x$. < $\longrightarrow 1^\infty$, indet. form so... >

So $\ln y = x \ln\left(1+\frac{c}{x}\right) = \dfrac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}}$.

$$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{c}{x}} \cdot D_x \frac{c}{x}}{D_x \frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{x+c}{x}}(c) = \lim_{x\to\infty} \frac{cx}{x+c} = c$$

$$\Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^c$$