Prof. Girardi, Math 142, Spring 2013, Final Exam.

Indication of Solutions.

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1. $\int_0^5 \frac{1}{x^2+25}\,dx = \frac{1}{5}\tan^{-1}\frac{x}{5}\Big|_{x=0}^{x=5} = \frac{1}{5}\left[\tan^{-1}1 - \tan^{-1}0\right]$

$= \frac{1}{5}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{20}.$

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2. $\int \frac{x}{x^2+25}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+25| + C.$

$\boxed{\begin{array}{l} u = x^2+25 \\ du = 2x\,dx \end{array}}$

So $\int_0^5 \frac{x\,dx}{x^2+25} = \frac{1}{2}\ln|x^2+25|\Big|_0^5$

$= \frac{1}{2}\left[\ln 50 - \ln 25\right]$

$= \frac{1}{2}\ln\frac{50}{25} = \frac{1}{2}\ln 2 = \ln 2^{1/2} = \ln\sqrt{2}.$

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3. $\int \frac{x^2}{x^2+25}\,dx = \int\left[1 - 5^2\frac{1}{x^2+5^2}\right]dx = x - 5^2\cdot\frac{1}{5}\tan^{-1}\frac{x}{5} + C = x - 5\tan^{-1}\frac{x}{5} + C$

do not have (strictly) bigger bottoms so need to do long division, which we "fake"

$\frac{x^2}{x^2+25} = \frac{x^2+25}{x^2+25} + \frac{-25}{x^2+25} = 1 - 25\frac{1}{x^2+25}.$

So $\int_0^5 \frac{x^2}{x^2+25}\,dx = \left(x - 5\tan^{-1}\frac{x}{5}\right)\Big|_{x=0}^{x=5} = \left[5 - 5\tan^{-1}1\right] - \left[0 - 5\tan^{-1}0\right]$

$= \left[5 - 5\left(\frac{\pi}{4}\right)\right] - [0] = \frac{20-5\pi}{4}$

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4. $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$

see textbook, § 7.4, Example 8, p 480. Do with PFD.

$= \ln\left(\frac{|x|}{(x^2+1)^{1/2}}\right) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C.$

5. $\int \ln x \, dx = x \ln x - x + C$. See textbook, §7.1, Example 2, p. 454.

So $\int_1^e \ln x \, dx = [x \ln x - x] \Big|_1^e = [e \ln e - e] - [\ln 1 - 1] = [e - e] - [-1] = 1$.

6. $\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C = e^x (x^2 - 2x + 2) + C$.

See textbook, §7.1, example 3, p. 455.

So $\int_0^1 x^2 e^x \, dx = [e^1 (1 - 2 + 2)] - [e^0 (0 - 0 + 2)]$

$= [e \quad (1) \quad] - [1 \quad (2)] = \boxed{e - 2}$.

7. $\int e^x \sin x \, dx = \dfrac{e^x (\sin x - \cos x)}{2} + C$ (see textbook, §7.1, Example 4, p. 455-6)

So $\int_0^{\pi/2} e^x \sin x \, dx = \dfrac{e^{\pi/2} (\sin \frac{\pi}{2} - \cos \frac{\pi}{2})}{2} - \dfrac{e^0 (\sin 0 - \cos 0)}{2}$

$= \dfrac{e^{\pi/2} \, (1 - 0)}{2} - \dfrac{1 \, (0 - 1)}{2} = \dfrac{e^{\pi/2}}{2} + \dfrac{1}{2}$.

$= \frac{1}{2} \left(1 + e^{\pi/2} \right)$

8. $\int \sin^2 x \, dx = \left[\frac{x}{2} - \frac{1}{4} \sin(2x) \right] + C$. (See textbook, §7.2, Example 3, p 461)

So $\int_0^{\pi/4} \sin^2 x \, dx = \left[\frac{1}{2} \left(\frac{\pi}{4} \right) - \frac{1}{4} \sin \frac{\pi}{2} \right] - \left[\frac{0}{2} - \frac{1}{4} \underbrace{\sin 0}_{=0} \right] = \boxed{\frac{\pi}{8} - \frac{1}{4}}$

9. $\int \dfrac{1}{[x^2 + 2x + 2]^2} \, dx = \int \dfrac{1}{[(x+1)^2 + 1]^2} \, dx = \int \dfrac{\sec^2 \theta}{(\sec^2 \theta)^2} \, d\theta = \int \cos^2 \theta \, d\theta$

$= \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta = \frac{1}{2} \left[\theta + \frac{\sin(2\theta)}{2} \right] + C$

$= \frac{1}{2} \theta + \frac{1}{2} \dfrac{2 \cos \theta \sin \theta}{2} + C = \frac{1}{2} \theta + \frac{1}{2} \cos \theta \sin \theta + C$

$= \frac{1}{2} \arctan(x+1) + \frac{1}{2} \cdot \dfrac{x+1}{x^2 + 2x + 2} + C$

$x + 1 = \tan \theta$

$dx = \sec^2 \theta \, d\theta$

$(x+1)^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta$

$\sqrt{(x+1)^2 + 1^2} = \sqrt{x^2 + 2x + 2}$

$2 \rightarrow$

So
$$\int_{-1}^{0} \frac{dx}{(x^2+2x+2)^2} = \left(\frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\,\frac{x+1}{x^2+2x+2}\right)\Bigg|_{x=-1}^{x=0}$$

$$= \left[\frac{1}{2}\underbrace{\tan^{-1}(1)}_{=\frac{\pi}{4}} + \frac{1}{2}\,\frac{1}{2}\right] - \left[\frac{1}{2}\underbrace{\tan^{-1}(0)}_{=0} + \frac{1}{2}\,0\right] = \frac{\pi}{8} + \frac{1}{4}$$

10. $\displaystyle\int \frac{-2}{(x-11)^3}\,dx = \int -2(x-11)^{-3}\,dx = \frac{-2(x-11)^{-2}}{-2} + C = \frac{1}{(x-11)^2} + C$

$$\int_{10}^{12} \frac{-2}{(x-11)^3}\,dx = \left[\lim_{a\to 11^-}\int_{10}^{a}\frac{-2}{(x-11)^3}\,dx\right] + \left[\lim_{b\to 11^+}\int_{b}^{12}\frac{-2}{(x-11)^3}\,dx\right]$$

$$= \left[\lim_{a\to 11^-}\frac{1}{(x-11)^2}\Bigg|_{x=10}^{x=a}\right] + \left[\lim_{b\to 11^+}\frac{1}{(x-11)^2}\Bigg|_{x=b}^{x=12}\right]$$

$$= \left[\lim_{a\to 11^-}\left(\frac{1}{(a-11)^2} - 1\right)\right] + \left[\lim_{b\to 11^+}\left(1 - \frac{1}{(b-11)^2}\right)\right]$$

$$= \left[\lim_{a\to 11^-}\frac{1}{(a-11)^2}\right] - 1 + 1 - \left[\lim_{b\to 11^+}\frac{1}{(b-11)^2}\right]$$

$$= \left[\lim_{a\to 11^-}\frac{1}{(a-11)^2}\right] - \left[\lim_{b\to 11^+}\frac{1}{(b-11)^2}\right]$$

$$\boxed{\infty \quad - \quad \infty}$$

→ indeterminate form, DNE.

The region R for problems 11–14

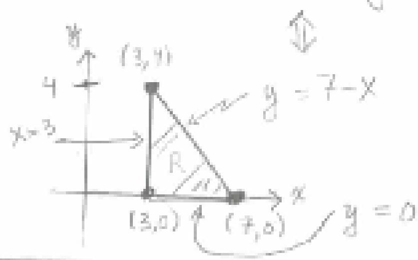
Let R be the region in the first quadrant enclosed by

$$y = 7 - x \quad \text{and} \quad y = 0 \quad \text{and} \quad x = 3 .$$

Note that R is the triangle with vertices: (3,0) and (3,4) and (7,0).

$x = 7 - y$

$\Updownarrow$



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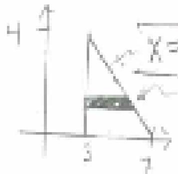
11. 

$\boxed{x=7-y}$ With respect to $y \Rightarrow$ partition y axis & all expressed in terms of y

Area of typical rectangle $=$ (height)(base) $= \left[(7-y)-3\right]\Delta y$

$$\text{Area of } R = \int_{y=0}^{y=4} \left[(7-y)-3\right] dy$$

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11. Let' check answer since we know formula for area of a triangle.

$$\int_0^4 \left[(7-y)-3\right] dy = \int_0^4 (4-y)\,dy = \left(4y - \frac{y^2}{2}\right)\Big|_0^4 = (16-8) - 0 = 8,$$

and area of triangle $= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(7-3)(4-0) = \frac{1}{\cancel{2}}\,\cancel{4}^{2}\cdot 4 = 8.$

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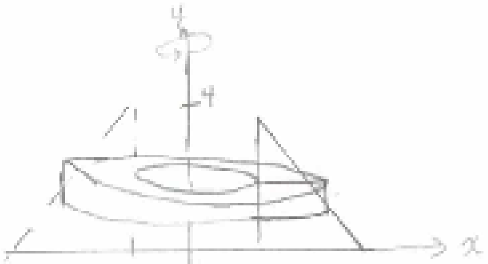
12 

wrt $y \Rightarrow$ partition y-axis & all expressed in terms of y

$\Downarrow$ rotate abt y-axis

Disk or washer

$\mapsto$ has hole

Volume of typical washer $= \pi\left[(\text{big radius})^2 - (\text{little radius})^2\right](\text{height})$

$$= \pi\left[(7-y)^2 - (3)^2\right]\Delta y$$

$$\text{Volume of solid} = \pi \underbrace{\int_{y=0}^{y=4} \left[(7-y)^2 - (3)^2\right] dy}$$

Check: Basic calculus gives $= \dfrac{208\pi}{3}$

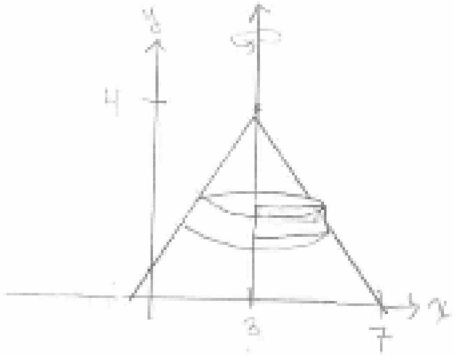
Volume cone $= \frac{1}{3}\pi r^2 h$
Volume Cylinder $= \pi r^2 h$

h = height
r = radius of base

Volume $=$ Volume of cone $\left\langle \begin{matrix} h=7 \\ r=7 \end{matrix} \right.$ $-$ Voume of cone $\left\langle \begin{matrix} h=3 \\ r=3 \end{matrix} \right.$ $-$ Volume cylinder $\left\langle \begin{matrix} h=4 \\ r=3 \end{matrix} \right.$

$$= \pi\left[\frac{1}{3}7^2\cdot 7 - \frac{1}{3}3^2\cdot 3 - 3^2\cdot 4\right] = \frac{\pi}{3}\left[343 - 27 - 108\right] = \frac{208\pi}{3}.$$

13.



wrt y ⇒ partition y-axis

⇓ rotate abt a line parallel to y-axis

Disk or Washer

↳ no hole

Volume of typical disk = $\pi(\text{radius})^2(\text{height})$

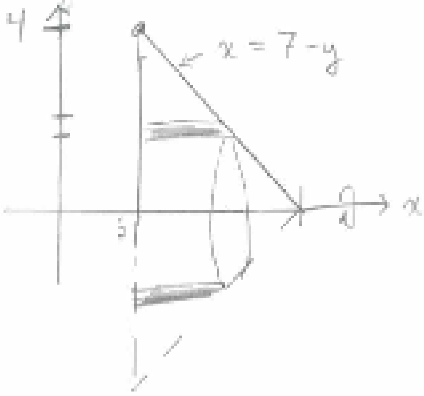
$= \pi\left((7-y)-3\right)^2(\Delta y)$.

$$\text{Volume of solid} = \underbrace{\pi \int_{y=0}^{y=4} \left((7-y)-3\right)^2 \, dy}_{=\frac{64\pi}{3}}$$

Check: Basic calculus gives $= \frac{64\pi}{3}$

Volume of cone $\left\langle \begin{matrix} h = 4 \\ r = 7-3 = 4 \end{matrix} \right. = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 4^2 \cdot 4 = \frac{64\pi}{3}$.

14.



wrt y ⇒ partition y-axis

⇓ rotate abt x-axis

Shell Method

Volume of typical shell = $2\pi(\text{avg radius})(\text{height})(\text{thickness})$

$= 2\pi(y)\left[(7-y)-3\right](\Delta y)$

$$\text{Volume of solid} = \underbrace{2\pi \int_{y=0}^{y=4} y\left[(7-y)-3\right] dy}$$

Check:

Basic calculus gives $= \frac{64\pi}{3}$

Volume of cone w/ height 4 & radius of base 4

$= \frac{\pi}{3} \cdot 4 \cdot 4^2 = \frac{64\pi}{3}$

15. $r = \tan\theta \sec\theta \iff r = \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} \iff r = \frac{\sin\theta}{\cos^2\theta}$

$\overset{(*)}{\Longrightarrow} r\cos^2\theta = \sin\theta \iff (r\cos\theta)^2 = r\sin\theta \iff \boxed{x^2 = y}$

Note that $\overset{(*)}{\Longrightarrow}$ can be reversed to $\Leftarrow$

Since if $r\cos^2\theta = \sin\theta$ then $\cos^2\theta \neq 0$.

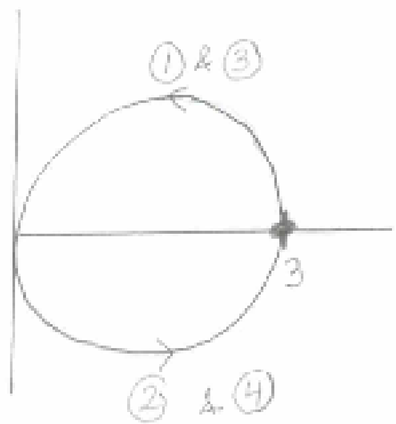
Indeed if $r\cos^2\theta = \sin\theta$ and $\cos^2\theta = 0$

then $0 = \sin\theta$ so $\cos^2\theta + \sin^2\theta = 0$,

which cannot be.

16. $r = 3\cos\theta$, $\frac{1}{4}$ (period of $\cos\theta$) $= \frac{1}{4}(2\pi) = \frac{\pi}{2}$.

θ	$r = 3\cos\theta$
$0 \xrightarrow{1} \frac{\pi}{2}$	$3 \to 0$
$\frac{\pi}{2} \xrightarrow{2} \pi$	$0 \to -3$
$\pi \xrightarrow{3} \frac{3\pi}{2}$	$-3 \to 0$
$\frac{3\pi}{2} \xrightarrow{4} 2\pi$	$0 \to 3$



$$A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta = \frac{1}{2}\int_0^\pi [3\cos\theta]^2\,d\theta$$

17. $\frac{\sqrt{25n^3 + 4n^2 + n - 5}}{7n^{3/2} + 6n - 1}$ $\boxed{\div \text{ num. \& dem. by } n^{3/2} = \sqrt{n^3}}$ $\quad \frac{\sqrt{25 + \frac{4}{n} + \frac{1}{n^2} - \frac{5}{n^3}}}{7 + \frac{6}{n^{1/2}} - \frac{1}{n^{3/2}}} \xrightarrow{n\to\infty} \frac{\sqrt{25}}{7} = \frac{5}{7}.$

18 Want $\sum_{n=2}^{\infty} r^n = \frac{1}{12}$. $\boxed{\text{We know that we \underline{must} have } |r|<1 \text{ (geom. series)}}$

Let $S_N = \sum_{n=2}^{N} r^n$, So

$$1\, S_N = r^2 + r^3 + r^4 + \dots + r^N$$

subtract $\quad r\, S_N = \quad r^3 + r^4 + \dots + r^N + r^{N+1}$

$$(1-r)\, S_N = r^2 - r^{N+1}$$

$$S_N \underset{=}{\overset{r\neq 1}{=}} \frac{r^2 - r^{N+1}}{1-r} \xrightarrow[\text{if } |r|<1]{N\to\infty} \frac{r^2}{1-r}$$

So want $r \in \mathbb{R}$ so that $|r|<1$ and $\frac{r^2}{1-r} = \frac{1}{12}$

$$\frac{r^2}{1-r} = \frac{1}{12} \iff 12r^2 = 1-r \iff 12r^2 + r - 1 = 0$$

$$\iff r = \frac{-1 \pm \sqrt{1^2 - 4(12)(-1)}}{2(12)} = \frac{-1 \pm \sqrt{1+48}}{2\cdot 12} = \frac{-1\pm 7}{2\cdot 12}$$

$$= \begin{cases} \frac{-1-7}{2\cdot 12} = \frac{-8}{2\cdot 12} = -\frac{2\cdot 4}{2\cdot 4\cdot 3} = -\frac{1}{3} \\ \frac{-1+7}{2\cdot 12} = \frac{6}{2\cdot 6\cdot 2} = \frac{1}{4} \end{cases}$$

19. Know (A) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p=1$, $p \le 1$) (or, also, harmonic series)

(B) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (just apply the AST).

Now, look at the choices.

20 $\frac{1}{\sqrt{(n+2)(n+7)}} \overset{n \text{ big}}{\sim} \frac{1}{\sqrt{n\cdot n}} = \frac{1}{n}$. Now look at the choices.

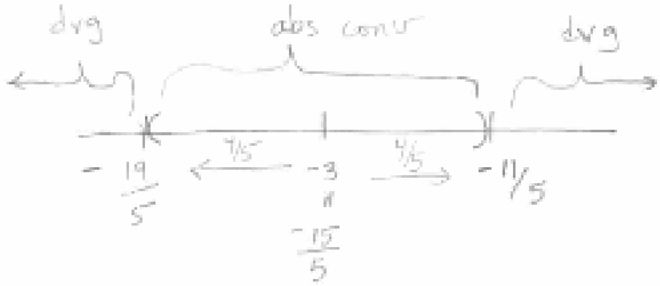
21. $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)!}{[3(n+1)]!} \cdot \dfrac{(3n)!}{n!} = \dfrac{(n!)(n+1)}{(n!)} \cdot \dfrac{(3n)!}{[(3n)!]\,(3n+1)(3n+2)(3n+3)}$

$= \dfrac{n+1}{(3n+1)(3n+2)(3n+3)} \xrightarrow{n\to\infty} 0.$

22. $\sum \dfrac{(5x+15)^n}{4^n} = \sum \dfrac{[5(x+3)]^n}{4^n} = \sum \left(\dfrac{5}{4}\right)^n (x - -3)^n. \Rightarrow$ center $= -3$

$\left[\left|\dfrac{(5x+15)^n}{4^n}\right|\right]^{1/n} = \left|\dfrac{5x+15}{4}\right| = \dfrac{5}{4}\,|x+3| \xrightarrow{n\to\infty} \dfrac{5}{4}\,|x+3|$

$\dfrac{5}{4}\,|x+3| < 1 \iff |x+3| < \dfrac{4}{5} \Rightarrow$ rad. of conv. is $\dfrac{4}{5}$



Check endpts

$x = -11/5$: $\sum \dfrac{(5x+15)^n}{4^n} = \sum \dfrac{4^n}{4^n} = \sum_{n=1}^{\infty} 1$ dvg (to ∞)

$x = -19/5$: $\sum \dfrac{(5x+15)^n}{4^n} = \sum \dfrac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n$ dvg (osc.)

23. $\ln(10-x) = \ln(1 + (9-x)) \overset{(*)}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{(9-x)^n}{n} = \rightarrow$

$\rightarrow = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{((-1)(x-9))^n}{n} = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(-1)^n (x-9)^n}{n}$

$= \sum_{n=1}^{\infty} (-1)^{2n+1} \dfrac{(x-9)^n}{n} = \sum_{n=1}^{\infty} \dfrac{-1}{n}(x-9)^n.$

valid $(*)$ $\quad -1 < 9-x \le 1 \iff -1 \le x-9 < 1 \iff 8 \le x < 10.$

24. The computations below show that the 3^{rd} order Taylor polynomial, about the center $x_0 = 1$, for the function $f(x) = x^5 - x^2 + 5$ is $p_3(x) = 5 + 3(x-1) + 9(x-1)^2 + 10(x-1)^3$.

we were given $x_0 = 1$			
n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$\frac{f^{(n)}(x_0)}{n!}$
0	$x^5 - x^2 + 5$	5	$\frac{5}{0!} = \frac{5}{1} = 5$
1	$5x^4 - 2x$	$5 - 2 = 3$	$\frac{3}{1!} = \frac{3}{1} = 3$
2	$5 \cdot 4x^3 - 2$	$20 - 2 = 18$	$\frac{18}{2!} = \frac{18}{2} = 9$
3	$5 \cdot 4 \cdot 3$	$(5)(4)(3)$	$\frac{(5)(4)(3)}{3!} = \frac{(5)(4)(3)}{(3)(2)} = \frac{(5)(4)}{2} = 10$

25. Consider the function $f(x) = e^{-x}$ as well as the interval $(7, 9)$.
The 5^{th} order Taylor polynomial of $y = f(x)$ about the center $x_0 = 0$ is

$$P_5(x) \ = \ \sum_{n=0}^{5} \frac{(-x)^n}{n!} \ = \ 1 \ - \ x \ + \ \frac{x^2}{2!} \ - \ \frac{x^3}{3!} \ + \ \frac{x^4}{4!} \ - \ \frac{x^5}{5!} \, .$$

The 5^{th} order Remainder term $R_5(x)$ is defined by $R_5(x) = f(x) - P_5(x)$ and so $f(x) \approx P_5(x)$ where the approximation is within an error of $|R_5(x)|$. Using Taylor's (BIG) Theorem, find a good upper bound for $|R_5(x)|$ that is valid for each $x \in (7, 9)$.
By Taylor's Remainder Theorem, for each $x \in (7, 9)$, there exists c between x and 0 so that

$$R_5(x) = \frac{f^{(6)}(c) \ (x-0)^6}{6!} \, .$$

Note that if $x \in (7, 9)$ and c is between x and 0, then $c \in (0, 9)$. So for each $x \in (7, 9)$,

$$|R_5(x)| = \left| \frac{f^{(6)}(c) \ (x-0)^6}{6!} \right| = \frac{\left| f^{(6)}(c) \right| \ |x|^6}{6!} = \frac{e^{-c} \ |x|^6}{6!} \le \frac{e^{-c} \ 9^6}{6!} \le \frac{e^{-0} \ 9^6}{6!} = \frac{9^6}{6!} \, .$$

If you prefer, you can also think of the above line as:

$$|R_5(x)| = \left| \frac{f^{(6)}(c) \ (x-0)^6}{6!} \right| = \frac{\left| f^{(6)}(c) \right| \ |x|^6}{6!} = \frac{e^{-c} \ |x|^6}{6!} = \frac{|x|^6}{e^c \ 6!} \le \frac{9^6}{e^c \ 6!} \le \frac{9^6}{e^0 \ 6!} = \frac{9^6}{6!} \, .$$