# Final Review

## Integration by Parts

Idea: $\int u\,dv = uv - \int v\,du$, where $u$ and $v$ are functions (most likely of $x$) and where $du$ and $dv$ denote the derivatives of $u$ and $v$ respectively (with respect to $x$).

Ex: $\int \ln x\,dx$

Let $u = \ln x$ and $dv = dx$. Then $du = 1/x\,dx$ and $v = x$. So

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = (\ln x)x - \int x\left(\tfrac{1}{x}\,dx\right)$$

$$= x\ln x - \int dx = x\ln x - x + C.$$

Keep in mind, $\int u\,dv$ must cover (multiplicatively) everything in the integral. Don't leave any bits out! Also, you cannot split portions of the function located under an inner function, so if something is under a square root / all being squared / in a sine / etc, it needs to stay together!

Ex: $\int xe^x \, dx$

$u = x \quad dv = e^x \, dx$
$du = dx \quad v = e^x$ gives

$$\int xe^x \, dx = \int u \, dv = uv - \int v \, du = xe^x - \int e^x \, dx$$

$$= xe^x - e^x + C.$$

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Ex: Integration by parts may be applied multiple times, as in this example.
Take $\int e^x \sin x \, dx$

Let $u = \sin x \quad dv = e^x \, dx$
$du = \cos x \, dx \quad v = e^x$

So $\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \cos x \, dx$.

Consider $\int e^x \cos x \, dx$, and let $\begin{array}{ll} u = \cos x & dv = e^x \, dx \\ du = -\sin x \, dx & v = e^x \end{array}$

Then $\int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \cos x - \int e^x (-\sin x) \, dx$

$$= e^x \cos x + \int e^x \sin x \, dx.$$

Hence, $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx\right)$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx.$$

So, $2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$, and $\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C$.

# Trig Integrals

The number one thing to know for trig integrals are all your trig identities. Once you have those, solving an integral usually amounts to making an appropriate u-substitution or by applying integration by parts.

Important identities (especially popular ones are $\star$-d):

$$\star \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\star \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\star \quad \sin^2 x + \cos^2 x = 1$$

Those two follow from the first.

$$\begin{cases} \tan^2 x + 1 = \sec^2 x \\ 1 + \cot^2 x = \csc^2 x \end{cases}$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

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Ex: $$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx$$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \cos^2(2x)\,dx = \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}\,dx$$

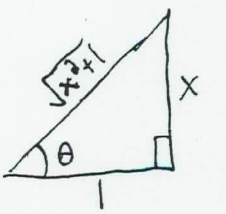
$$= \frac{1}{8}\int 2 + 4\cos 2x + 1 + \cos 4x\,dx = \frac{1}{8}\int 3 + 4\cos 2x + \cos 4x\,dx$$

$$= 1/8\left(3x + 2\sin 2x + \frac{1}{4}\sin 4x\right) + C$$

# Trig Substitution

Idea: replace portions of a difficult integral with trig things from an appropriately defined triangle.

Ex: $\displaystyle\int \frac{1}{x^2+1}\,dx$




Define your triangle as at the left. Note that the Pythagorean Theorem holds for this triangle (as it should for any right triangle).

$$\tan\theta = x \quad\Rightarrow\quad dx = \sec^2\theta\,d\theta$$

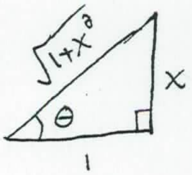
$$\cos\theta = \frac{1}{\sqrt{x^2+1}} \quad\Rightarrow\quad \cos^2\theta = \frac{1}{x^2+1}$$

Hence, $\displaystyle\int \frac{1}{x^2+1}\,dx = \int \cos^2\theta\,(\sec^2\theta)\,d\theta = \int d\theta = \theta + C$

However, $\tan\theta = x \Rightarrow \theta = \arctan x$. Therefore,

$$\int \frac{1}{x^2+1}\,dx = \theta + C = \arctan x + C.$$

Ex: $\displaystyle\int \frac{1}{x\sqrt{1+x^2}}\,dx$




$$\tan\theta = x \Rightarrow \sec^2\theta \overset{d\theta}{\downarrow} = dx$$
$$\cot\theta = 1/x$$
$$\cos\theta = 1/\sqrt{1+x^2}$$

Hence, $\displaystyle\int \frac{1}{x\sqrt{1+x^2}}\,dx = \int \frac{1}{x}\cdot\frac{1}{\sqrt{1+x^2}}\,dx = \int \cot\theta\,\cos\theta\,\sec^2\theta\,d\theta$

$$= \int \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{1}{\cos^2\theta}\,d\theta = \int \frac{1}{\sin\theta}\,d\theta$$

$$= \int \csc\theta\,d\theta = -\ln\left|\csc\theta + \cot\theta\right| + C$$

$$= -\ln\left|\frac{1}{x}\sqrt{1+x^2} + \frac{1}{x}\right| + C = -\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| + C$$

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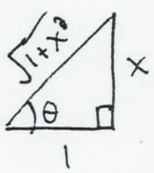
Ex. $\displaystyle\int \frac{x^2}{\sqrt{1+x^2}}\,dx$




$$\tan\theta = x \Rightarrow dx = \sec^2\theta\,d\theta$$
$$\Rightarrow x^2 = \tan^2\theta$$

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \int x^2\left(\frac{1}{\sqrt{1+x^2}}\right)dx = \int \tan^2\theta\,\cos\theta\,\sec^2\theta\,d\theta = \int \tan^2\theta\,\sec\theta\,d\theta$$

$$\int \sec^3\theta\,d\theta - \int\sec\theta\,d\theta = \int(\sec^2\theta - 1)\sec\theta\,d\theta = \int \tan^2\theta\,\sec\theta\,d\theta = \int \overset{u}{\tan\theta}\,(\overset{dv}{\sec\theta\tan\theta})\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta$$

* Then solve for $\int\sec^3\theta\,d\theta$, resubstitute into $\sec\theta\tan\theta - \int\sec^3\theta\,d\theta$ for $x$.

# Partial Fraction Decomposition

Idea: factor the denominator, and split the fraction up into additive parts so that the numerators have degree 1 less than the denominators. For instance:

| Original Function | Decomposes like |
| --- | --- |
| $\dfrac{1}{(x-1)(x+1)}$ | $\dfrac{A}{x-1} + \dfrac{B}{x+1}$ |
| $\dfrac{1}{(x^2+1)(x-1)}$ | $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1}$ |
| $\dfrac{1}{(x^3+x+1)(x^2+1)(x+1)}$ | $\dfrac{Ax^2+Bx+C}{x^3+x+1} + \dfrac{Dx+E}{x^2+1} + \dfrac{F}{x+1}$ |

In the event that a root is repeated, make a new summand for each degree leading up to the number of repetitions of the root. For instance:

| Original function | Decomposes like |
| --- | --- |
| $\dfrac{1}{(x-1)^2(x+1)}$ | $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$ |
| $\dfrac{1}{(x-1)^3(x+1)}$ | $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{D}{x+1}$ |
| $\dfrac{1}{(x^2+1)^2(x+1)}$ | $\dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2} + \dfrac{E}{x+1}$ |

In the context of integration, this technique can be used to make integration easier or to make it possible for a given function.

Ex: $\int \frac{1}{x^2-1}\,dx$

Well, $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$. Set $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$.

Then $A(x+1) + B(x-1) = 1$. Letting $x=1$ gives $A = 1/2$. Letting $x=-1$ gives $B = -1/2$. So $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}$.

Therefore,

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$$

Ex: $\int \frac{dx}{(x^2+1)(x+1)}$

After PFD, $\int \frac{dx}{(x^2+1)(x+1)} = \int \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} + \frac{1/2}{x+1}\,dx = \frac{1}{2}\int \frac{1-x}{x^2+1} + \frac{1}{x+1}\,dx$

$$= \frac{1}{2}\int \frac{-x}{x^2+1} + \frac{1}{x^2+1} + \frac{1}{x+1}\,dx = \frac{1}{2}\int \frac{-1}{2}\left(\frac{2x}{x^2+1}\right) + \frac{1}{x^2+1} + \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\left(\frac{-1}{2}\ln(x^2+1) + \arctan(x) + \ln|x+1|\right) + C.$$

# Improper Integrals

Idea: Integrating over discontinuities or off to infinity. To be technically accurate, we will need to combine integration techniques with limit techniques.

Ex: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{x}\Big|_b^1 = \lim_{b\to\infty}\left(1 - 1/b\right) = 1 - 0 = \underline{1}.$$

Ex: $\displaystyle\int_0^\infty xe^{-x}\,dx$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

gives $\displaystyle\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C.$

Hence, $\displaystyle\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}\int_0^b xe^{-x}dx = \lim_{b\to\infty}\left(-xe^{-x} - e^{-x}\Big|_0^b\right)$

$$= \lim_{b\to\infty}\left(xe^{-x} + e^{-x}\Big|_b^0\right) = 1 - \lim_{b\to\infty}\left(\frac{b}{e^b} + \frac{1}{e^b}\right) \overset{H}{=} \underline{1} - \lim_{b\to\infty}\left(\frac{1}{e^b}\right)$$

$$= \underline{1}.$$

## Sequences

Idea: You have an infinite set of numbers that may or may not converge to a single number at infinity. To take such a limit, you will use limit notation, restricting the variable over which the limit is being taken to the integers.

Note: If the limit of $f(x)$ exists and is $L$, then the limit of the sequence $\{f(n)\}$ $(n = 1, 2, 3, \dots)$ exists and is also $L$. The converse does not hold in general.

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## Integral Test

Idea: If $\int_a^\infty f(x)\,dx$ converges, then $\sum_{n=a}^{\infty} f(n)$ converges (not necessarily to the same value). Also, if $\int_a^\infty f(x)\,dx$ diverges, then $\sum_{n=a}^{\infty} f(n)$ diverges.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\int_1^\infty \frac{1}{x}\,dx = \ln|x| \Big|_1^\infty \text{ "="} \infty - 1 \text{, diverges} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

# Direct Comparison Test

Idea: A series that is less than a convergent series (for sufficiently large n) is convergent. A series that is greater than a divergent series (for sufficiently large n) is divergent. The preceding statements both hold so long as the terms of the series are positive.

Ex: $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$

Since $\frac{n-1}{n^2\sqrt{n}} \leq \frac{n}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series, $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ converges.

Ex: $\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$

Since $\left(\frac{\sin n}{n}\right)^2 = \frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series, $\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$ converges.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n!}$

Since $\left(\frac{1}{n!}\right) \leq \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by geometric series, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

# Limit Comparison Test

Idea: If $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$ then both series converge or both series diverge.

Ex: $$\sum_{n=9}^{\infty} \frac{1}{n^{3/2}-2n-6}$$

Consider $\sum_{n=9}^{\infty} \frac{1}{n^{3/2}}$, which converges by p-series. Then we have:

$$\lim_{n\to\infty} \left( \frac{\frac{1}{n^{3/2}-2n-6}}{\frac{1}{n^{3/2}}} \right) = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}-2n-6} = \lim_{n\to\infty} \frac{1}{1-2n^{-1/2}-6n^{-3/2}} = 1 > 0.$$

Therefore, $\sum_{n=9}^{\infty} \frac{1}{n^{3/2}-2n-6}$ converges.

Ex: $$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges by p-series. Then we have

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{k\to 0} \frac{\sin(k)}{k} \overset{\text{H}}{=} \lim_{k\to 0} \cos(k) = 1 > 0.$$

Therefore, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges.

## Alternating Series Test

Idea: If a sum is alternating and the terms have limit zero (and are decreasing), the series converges.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

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## Ratio Test

Idea: A series $\sum a_n$ converges if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ and diverges if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$. The test is indeterminate otherwise.

Note: use this test if you see factorials!

Ex: $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{100} \right| = \infty > 1. \text{ So } \sum_{n=1}^{\infty} \frac{n!}{100^n} \text{ diverges.}$$

Ex: $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

$$\lim_{n\to\infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{2n^2} \right| = \frac{1}{2} < 1, \text{ so } \sum_{n=1}^{\infty} \frac{n^2}{2^n} \text{ converges.}$$

# Root Test

Idea: A series $\sum a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ and diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$. The test is inconclusive otherwise.

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Ex: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1, \text{ so } \sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n \text{ converges.}$$

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Ex: $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1, \text{ so root test is inconclusive.}$$

However, noting that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$ shows the series diverges.

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## Power Series

Idea: Turn a regular function into an infinite sum.

Ex: $\frac{1}{1-x}$. Doing long division gives

$$\begin{array}{r|l} & 1 + x + x^2 + \ldots \\ \hline 1-x & 1 \\ & -(1-x) \\ \hline & x \\ & -(x-x^2) \\ \hline & x^2 \\ & \vdots \end{array}$$

Hence, $\displaystyle\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$

# Taylor Series

Idea: By picking a center c, we may write any function (in some radius of convergence about c) as an infinite sum using various ordered derivatives of the function evaluated at c.

The Taylor Series of $f(x)$ at $c$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Note: Power Series are a special case of Taylor Series in which finding the derivatives isn't necessary.

Double note: Much of the time, you can make your life easier by finding the Taylor series of a simpler function and then substituting for x.

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Ex: Find the Taylor Series and radius and interval of convergence for $\arctan x$, at $c=0$.

Note that $\arctan x = \displaystyle\int_0^x \frac{1}{t^2+1}\,dt$. Therefore, finding the Taylor Series for $\frac{1}{t^2+1}$ will put us in good shape. However, we already know the power series for $\displaystyle\frac{1}{t^2+1} = \frac{1}{1-(-t^2)} = \sum_{n=0}^{\infty}(-t^2)^n = \sum_{n=0}^{\infty}(-1)^n t^{2n}$. Since this power series is already at center $c=0$, we need only integrate to find the power series (and hence the Taylor Series) of $\arctan x$.

$$\arctan x = \int_0^x \frac{1}{t^2+1}\,dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{2n+1}\Bigg|_0^x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

↓ cont'd.

$\downarrow$ cont'd.

The radius of convergence can be obtained (in part) by the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+1}\cdot\frac{2n+1}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)x^{2n+3}}{(2n+3)x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)x^2}{2n+3}\right|$$

$$= x^2.$$

$$x^2 < 1 \Rightarrow -1 < x < 1$$

Now we need to check the endpoints.

$x = 1$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{2n+1}$, converges by AST.

$x = -1$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{-1}{2n+1}$, converges by AST.

Hence, the interval of convergence of the Taylor series is $[-1, 1]$, with radius $1$.

Ex: $f(x) = e^{-x^2} + \cos x$

Note: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. These can be verified by applying the definition for the Taylor series to each.

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$. Therefore,

$$e^{-x^2} + \cos x = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\right) + \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} x^{2n} + \frac{(-1)^n}{(2n)!} x^{2n}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n!} + \frac{1}{(2n)!}\right) x^{2n}$$

We apply ratio test for the interval of convergence:

$$\lim_{n\to\infty} \left| \frac{\left(\frac{1}{(n+1)!} + \frac{1}{(2(n+1))!}\right) x^{2(n+1)}}{\left(\frac{1}{n!} + \frac{1}{(2n)!}\right) x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} + \frac{1}{(2n+2)!}}{\frac{1}{n!} + \frac{1}{(2n)!}} x^2 \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(2n+2)! + (n+1)!}{(n+1)!\,(2n+2)!}}{\frac{(2n)! + n!}{n!\,(2n)!}} x^2 \right| = \lim_{n\to\infty} \left| \frac{(2n+2)! + (n+1)!}{(n+1)!\,(2n+2)!} \cdot \frac{n!\,(2n)!}{(2n)! + n!} x^2 \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)! + (n+1)!}{(n+1)(2n+2)(2n+1)\left((2n)! + (n)!\right)} x^2 \right| = \lim_{n\to\infty} \left| \frac{\text{degree } 2n+2 \text{ poly}}{\text{degree } 2n+3 \text{ poly}} x^2 \right|$$

$= 0 < 1$ $\forall x$. Therefore, the IOC is $(-\infty, \infty)$.

Ex: Calculate $e^{-0.2}$ correct to 5 decimal places.

We apply the Taylor Remainder Theorem. The error in the $n$th Taylor polynomial is less than or equal to

$$\max_{x_1, x_2 \in I} \left| \frac{f^{(n+1)}(x_1)}{(n+1)!} (x_2 - c)^{n+1} \right|,$$

where $I$ is the interval under consideration and $c$ is the center. In this case, the interval $[-0.2, 0]$ will suffice. Since $\frac{d^n}{dx^n} e^x = e^x$ for any $n$, we have

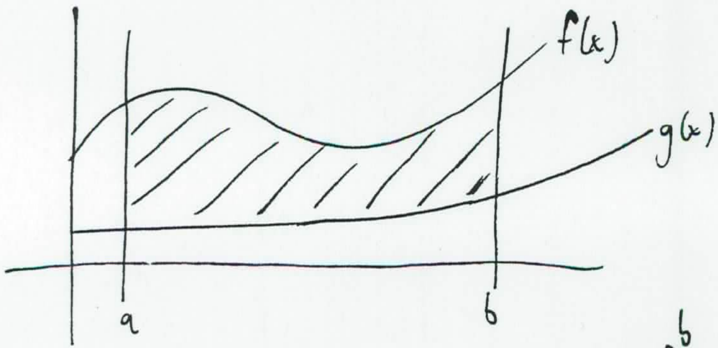
$$\text{Error} \leq \max_{x_1, x_2 \in [-0.2, 0]} \left| \frac{e^{x_1}}{(n+1)!} (x_2 - 0)^{n+1} \right|$$

Further, since $e^x$ increases on its whole domain, the error may be maximized as follows:

$$\text{Error} \leq \max_{x_1, x_2 \in [-0.2, 0]} \left| \frac{e^{x_1}}{(n+1)!} (x_2)^{n+1} \right| = \left| \frac{e^0}{(n+1)!} (-0.2)^{n+1} \right| = \frac{0.2^{n+1}}{(n+1)!}$$

Plugging in $n$s until $\frac{0.2^{n+1}}{(n+1)!} \leq 0.00001$ will finish it off.

# Area between curves




The shaded area is $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

or written more simply, $\int_a^b f(x) - g(x)\,dx$.

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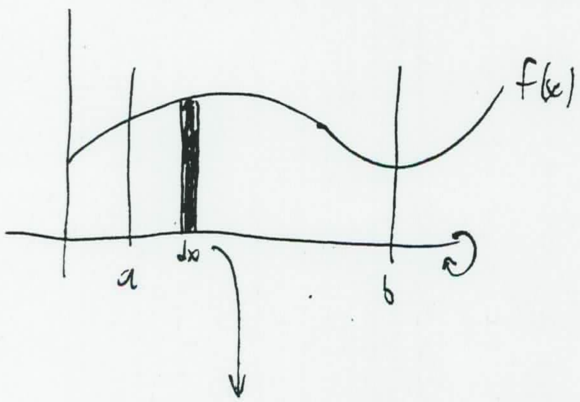
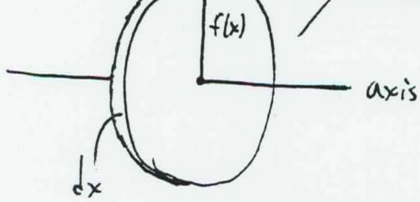
# Volumes of Revolution: Disk method



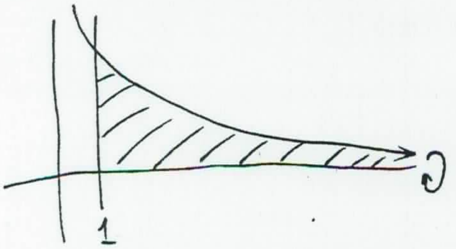

Idea: rotating f(x) about an axis, we seek the volume of the resulting solid.

Volume $= \pi (f(x))^2 dx$




By summing over an infinite number of such dx-disks, we obtain our formula: $\pi \int_a^b (f(x))^2\,dx$. (assuming the axis of rotation is the x-axis).

Ex: (Gabriel's Horn) Rotate $1/x$ about the x-axis from $x=1$ to $\infty$.




$$\pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2}\, dx$$

$$= \pi \int_1^\infty x^{-2}\, dx = \pi \left( -x^{-1} \Big|_1^\infty \right)$$

$$= \pi \left( x^{-1} \Big|_\infty^1 \right) = \pi(1-0) = \underline{\underline{\pi}}$$

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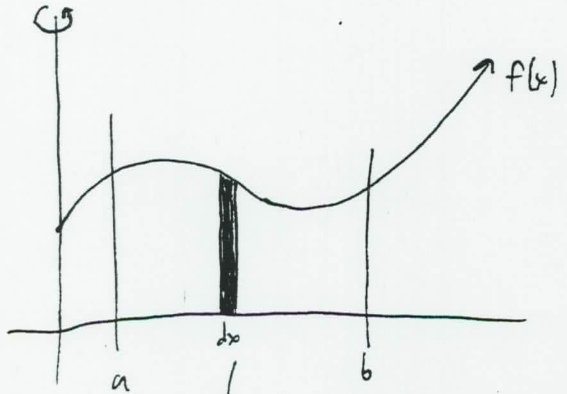
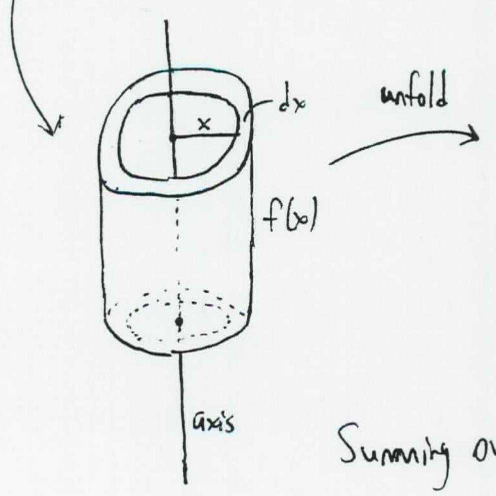
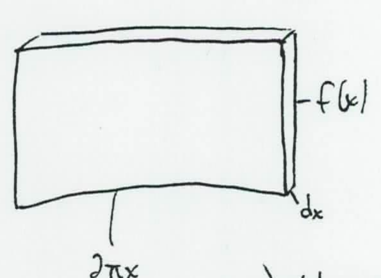
<u>Volumes of Revolution: Shell method</u>




Idea: Same as first section, but now we will piece the volume together in a different way.

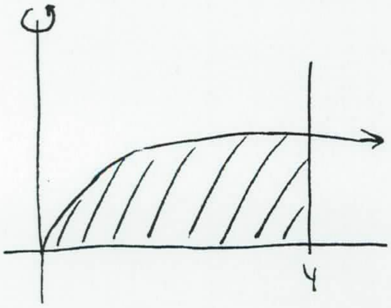



Volume $= 2\pi x f(x)\, dx$.

Summing over an infinite number of such dx-shells, we obtain the formula: $2\pi \int_a^b x f(x)\, dx$ (assuming the axis of rotation is the y-axis).

Ex: Rotate $\sqrt{x}$ about the $y$-axis from $x=0$ to $x=4$.




$$2\pi \int_0^4 x\sqrt{x}\,dx = 2\pi \int_0^4 x^{3/2}\,dx$$

$$= 2\pi \left( \frac{2x^{5/2}}{5} \Big|_0^4 \right)$$

$$= 2\pi \left( \frac{2}{5} \cdot 32 \right) = \frac{128}{5}\pi$$

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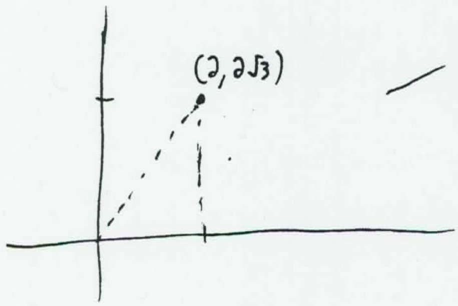
## Polar Coordinates

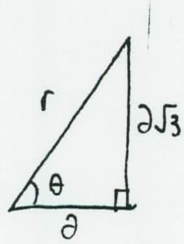
Idea: coordinatize the plane by using radius and angle.

Ex:




— put $(2, 2\sqrt{3})$ into polar form.

we obtain a triangle

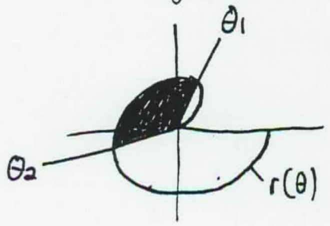



It happens to be a 30-60-90 triangle, so $\theta = 60° = \pi/3$ rads, and $r = 4$.

Hence, $(2, 2\sqrt{3}) \longmapsto (4, \pi/3)$ in polar.

# Polar area

Integration in polar will yield the area of the space between a polar function's edge and the origin over a theta-range, as below. However, correction factors need to be made to account for the circular method in which we are integrating.




$$\text{Area b/w } \theta_1 \text{ and } \theta_2 = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$$

Note of warning: polar functions really, REALLY like to cancel out their own areas. You will likely need to break up your integral to obtain the actual area of a region, especially if the function in question is rose-type. Use the symmetries of the function to your advantage to make sure your solution is correct.

* in the case of a rose: find the area of one petal and then multiply by the number of petals.