HAND IN PART

Prof. Girardi | Math 142 | Spring 2013 | 03.26.2013 | Exam 2

MARK BOX		
PROBLEM	POINTS	
0	30	
1	10	
2	10	
3	10	
4	10	
5–10	30	
%	100	

NAME: Sol'n

PIN: 17

INSTRUCTIONS

(1) The MARK BOX above indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(2) You may **not** use an electronic device, a calculator, books, personal notes.

(3) On Problem 0, fill in the blanks. As you were warned, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.

(4) For Problems 1–4, to receive credit you must:

- (a) **work in a logical fashion, show all your work, indicate your reasoning**; **no credit will be given for an answer that *just appears***; such explanations help with partial credit
- (b) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
- (c) if no such line/box is provided, then box your answer.

(5) Problems 5–10 are multiple choice.

- First, indicate (to yourself) your answers directly on the STATEMENT OF MULTIPLE CHOICE PROBLEMS part.
- Once finished with the multiple choice problems, go back to the HAND IN PART and indicate your answers on the table provided.
- Hand in the HAND IN PART. You can take the STATEMENT OF MULTIPLE CHOICE PROBLEMS part home with you (so you can check your answers once the solutions are posted).

(6) During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.

(7) This exam covers (from *Calculus* by Stewart, 6^{th} ed., ET): § 11.2 – 11.8 .

0. Fill-in-the blanks. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0a. **n^{th}-term test** for an arbitrary series $\sum a_n$.

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges.

0b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ < 1
- diverges if and only if $|r|$ ≥ 1

0c. ***p*-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p > 1
- diverges if and only if p ≤ 1

0d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

non increasing is also ok

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a decreasing function
- f is a continuous function .

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

0e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$. (Fill in the blanks with a_n and/or b_n.)

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum$ b_n converge, then $\sum$ a_n converge.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum$ b_n diverge, then $\sum$ a_n diverge.

0f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If 0 $< L <$ ∞, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

0g. Ratio and Root Tests for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$.

Let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.

- If ρ < 1 then $\sum a_n$ converges absolutely.
- If ρ > 1 then $\sum a_n$ diverges.
- If ρ $= 1$ then the test is inconclusive.

0h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

0i. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

0k. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

T F If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

T F If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges eg $\sum \frac{1}{n}$

T F If $S_N = \sum_{n=1}^{N} r^n$, then $S_N = \frac{r - r^{N+1}}{1-r}$, for when $r \neq 1$ since we don't like to divide by zero.

1. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=4}^{\infty} \frac{1}{n^{1.001}}$$

☒ absolutely convergent

☐ ~~conditionally convergent~~ b/c positive-term series.

☐ divergent

p-series

$p = 1.001 > 1$

so converges.

2. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3n^3 + 2n^2}$$

- [ ] absolutely convergent
- [ ] conditionally convergent
- [x] divergent

abs. conv?

$$\lim_{n \to \infty} \frac{n^3}{3n^3 + 2n^2} = \frac{1}{3} \neq 0$$

so by n^{th} term test for divg, $\sum \frac{n^3}{3n^3+2n^2}$ divg.

cond. conv?

$$\lim_{n \to \infty} (-1)^n \frac{n^3}{3n^3 + 2n^2} \quad \text{DNE} \neq 0.$$

osc.

so by n^{th} term test for divg., $\sum (-1)^n \frac{n^3}{3n^3+2n^2}$ divg

3. Let

$$a_n = \frac{n!}{(2n-1)!}$$

3a. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n(2n+1)} \quad \underline{\underline{\text{or}}} \quad \frac{n+1}{4n^2+2n}$$

3b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n-1)!}$$

a_n

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

$$a_{n+1} = \frac{(n+1)!}{(2(n+1)-1)!} = \frac{(n+1)!}{(2n+2-1)!} = \frac{(n+1)!}{(2n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2n+1)!} \cdot \frac{(2n-1)!}{n!} \quad \text{collect up like terms} \quad \frac{(n+1)!}{n!} \quad \frac{(2n-1)!}{(2n+1)!}$$

Cancellation heaven

$$= \frac{n!\,(n+1)}{n!} \quad \frac{(2n-1)!}{(2n-1)!\,(2n)(2n+1)} = \frac{n+1}{(2n)(2n+1)}$$

$$= \frac{n+1}{4n^2+2n} = \frac{\frac{1}{n} + \frac{1}{n^2}}{4 + \frac{2}{n}} \xrightarrow{n \to \infty} \frac{0+0}{4+0}$$

$\div$ by n highest power $= n^2$

$$= 0 = \rho < 1$$

By ratio Test, since $\rho < 1$, $\Sigma (-1)^n a_n$ is abs conv.

4. Consider the formal power series

$$\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}. \quad = \sum_{n=2}^{\infty} \frac{(x-0)^n}{(\ln n)^n}$$

center $= 0$

Hint 1: $\frac{x^n}{(\ln n)^n} = \left[\frac{x}{\ln n}\right]^n$ so would you rather use the $\boxed{\text{root test}}$ or the ratio test?

Hint 2: $\ln(a^r) = r\ln(a)$ but $(\ln(a))^r \neq r\ln(a)$

The center is $x_0 =$ ___0___ and the radius of convergence is $R =$ ___∞___. As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.

abs. conv.

0

Root Test

$$\lim_{n\to\infty} \left|\frac{x^n}{(\ln n)^n}\right|^{1/n} = \lim_{n\to\infty} \frac{|x|}{\ln n} = |x| \lim_{n\to\infty} \frac{1}{\ln n}$$

$$= |x| \cdot 0 \overset{\text{want}}{<} 1$$

true for all $x \in \mathbb{R}$

Let's see how the solution goes if we use the ratio test.

$$0 \le \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{[\ln(n+1)]^{n+1}} \cdot \frac{(\ln n)^n}{x^n}\right| = \frac{|x|^{n+1}}{|x|^n} \frac{(\ln n)^n}{[\ln(n+1)]^{n+1}}$$

$$= |x| \cdot \underbrace{\frac{(\ln n)^n}{[\ln(n+1)]^{n+1}}}_{\text{yuck - I do not want to deal with him, but can...}} = |x| \left[\frac{\ln n}{\ln(n+1)}\right]^n \cdot \frac{1}{\ln(n+1)}$$

$$\le |x| \cdot 1^n \cdot \frac{1}{\ln(n+1)} = |x| \frac{1}{\ln(n+1)} \xrightarrow{n\to\infty} |x| \cdot 0$$

$\ln n \le \ln(n+1)$

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS 5 – 10

Instructions.

- Indicate (by circling) your solution to each problem.
- You may choice up to 3 answers for each problem. The scoring is as follows. For a problem with precisely one answer marked and the answer is correct, 5 points. For a problem with precisely two answers marked, one of which is correct, 3 points. For a problem with precisely three answers marked, one of which is correct, 1 point. All other cases, 0 points.

Your Solutions						
PROBLEM						points
5	5a	5b	5c	5d	5e	
6	6a	6b	6c	6d	6e	
7	7a	7b	7c	7d	7e	
8	8a	8b	8c	8d	8e	
9	9a	9b	9c	9d	9e	
10	10a	10b	10c	10d	10e	

STATEMENT OF MULTIPLE CHOICE PROBLEMS

Instructions.

- Indicate (by circling) your solution to each problem on the Hand In Part of the exam.
- You may choice up to 3 answers for each problem. The scoring is as follows. For a problem with precisely one answer marked and the answer is correct, 5 points. For a problem with precisely two answers marked, one of which is correct, 3 points. For a problem with precisely three answers marked, one of which is correct, 1 point. All other cases, 0 points.

5. Evaluate

$$\sum_{n=17}^{\infty} 5\,\frac{(-2)^n}{3^{2n+1}}$$

a. $\left(\frac{7}{11}\right)\left(\frac{5}{3}\right)\left(\frac{-2}{9}\right)^{18}$ b. $\left(\frac{9}{11}\right)\left(\frac{-2}{9}\right)^{17}$ c. $\left(\frac{7}{11}\right)\left(\frac{5}{3}\right)\left(\frac{-2}{9}\right)^{17}$ d. $\left(\frac{9}{11}\right)\left(\frac{5}{3}\right)\left(\frac{-2}{9}\right)^{17}$ e. None of these

6. Evaluate

$$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$$

Hint: Telescoping Series, use PFD.

a. 2 b. $\frac{1}{2}$ c. 1 d. 4 e. None of these

7. Consider the following two series.

$$\text{Series A is} \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\text{Series B is} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\ .$$

a. both series converge absolutely b. both series diverge
c. series A converges conditionally and series B diverges
d. series A diverges and series B converges conditionally e. none of these

8. Consider the formal series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = (-1)^n \frac{(n+1)!}{(2n)!}$$

and let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$

a. $\sum_{n=1}^{\infty} a_n$ converges absolutely because $\rho = \frac{1}{2}$. b. $\sum_{n=1}^{\infty} a_n$ converges absolutely because $\rho = 0$.
c. $\rho = 1$ so the Ratio Test fails for $\sum_{n=1}^{\infty} a_n$ d. $\sum_{n=1}^{\infty} a_n$ diverges e. none of these

9. What is the LARGEST interval for which the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

is absolutely convergent?

a. $(1,5)$ b. $(-4,-2)$ c. $(-5,-1)$ d. $[-5,-1]$ e. none of these

10. Suppose that the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is 16. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{2n}$?

a. 256 b. 4 c. 1 d. 16 e. none of these

5. $$\sum_{n=17}^{\infty} 5\,\frac{(-2)^n}{3^{2n+1}} = \sum_{n=17}^{\infty} \frac{5}{3}\left(-\frac{2}{9}\right)^n$$

$$\overset{\text{see}}{\underset{\text{below}}{=}} \left(\frac{5}{3}\right)\left(\frac{9}{11}\right)\left(-\frac{2}{9}\right)^{17}$$

Let
$$1\, S_N = \left(-\frac{2}{9}\right)^{17} + \left(-\frac{2}{9}\right)^{18} + \dots + \left(-\frac{2}{9}\right)^{N}$$
$$-\frac{2}{9}\, S_N = \left(-\frac{2}{9}\right)^{18} + \dots + \left(-\frac{2}{9}\right)^{N} + \left(-\frac{2}{9}\right)^{N+1}$$

$$\underbrace{\left(1 - -\frac{2}{9}\right)} S_N = \left(-\frac{2}{9}\right)^{17} - \left(-\frac{2}{9}\right)^{N+1}$$

$$\frac{9}{9} + \frac{2}{9} = \frac{11}{9}$$

$$S_N = \frac{9}{11}\left[\left(-\frac{2}{9}\right)^{17} - \left(-\frac{2}{9}\right)^{N+1}\right] \xrightarrow{N \to \infty} \frac{9}{11}\left(-\frac{2}{9}\right)^{17}$$

6. $$\frac{4}{(4n-3)(4n+1)} \overset{\text{PFD}}{=} \frac{A}{4n-3} + \frac{B}{4n+1} = \frac{A(4n+1) + B(4n-3)}{(4n-3)(4n+1)}$$

$$\Rightarrow 4 = A(4n+1) + B(4n-3)$$

n^1 : $0 = 4A + 4B \Rightarrow A = -B \Rightarrow 4 = -B - 3B = -4B$

n^0 : $4 = A - 3B$

$$\Downarrow$$
$$B = -1$$
$$\Downarrow$$
$$A = -B = 1$$

$$S_N = \sum_{n=1}^{N} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{N} \left[\frac{1}{4n-3} + \frac{-1}{4n+1} \right]$$

$$= \frac{1}{1} + \cancel{\frac{-1}{5}} \quad \leftarrow n=1$$

$$+ \cancel{\frac{1}{5}} + \cancel{\frac{-1}{9}} \quad \leftarrow n=2$$

$$+ \cancel{\frac{1}{9}} + \cancel{\frac{-1}{13}} \quad \leftarrow n=3$$

$$+ \cancel{\frac{1}{13}} + \cancel{\frac{-1}{17}} \quad \leftarrow n=4$$

$\vdots$ yes ... we see the pattern & the cancellations

$$\cancel{\frac{1}{4N-3}} + \frac{-1}{4N+1} \quad \leftarrow n=N$$

$$= 1 + \frac{-1}{4N+1} \xrightarrow{N\to\infty} \boxed{1}.$$

7. $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. $\Big\{$ p-series, $p=1$, $p \le 1$ or harmonic series

$\sum \frac{(-1)^n}{n}$ conv. by AST since $\frac{1}{n} > \frac{1}{n+1}$

and $\lim_{n\to\infty} \frac{1}{n} = 0$

8. $a_n = (-1)^n \dfrac{(n+1)!}{(2n)!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{[(n+1)+1]!}{[2(n+1)]!} \cdot \frac{(2n)!}{(n+1)!}$$ < collect up like terms >

$$= \frac{(n+2)!}{(n+1)!} \quad \frac{(2n)!}{(2n+2)!}$$ < simplify … cancel >

$$= \frac{(n+1)!\,(n+2)}{(n+1)!} \quad \frac{(2n)!}{(2n)!\,(2n+1)(2n+2)}$$

$$= \frac{n+2}{(2n+1)(2n+2)} = \frac{n+2}{4n^2 + ?\,n^1 + ?\,n^0} \xrightarrow{n \to \infty} \underset{\rho}{\underset{|||}{0}} < 1.$$

abs. conv by ratio test.

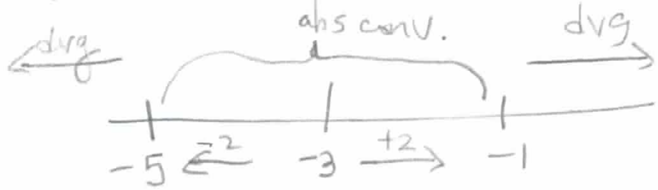
9. $$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n (x - {}^-3)^n}{4^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} (x - {}^-3)^n$$

center is $\underline{\underline{-3}}$.

root test

$$\lim_{n\to\infty} \left| \frac{1}{2^n} (x - {}^-3)^n \right|^{1/n} = \lim_{n\to\infty} \frac{1}{2} |x - {}^-3| = \frac{1}{2} |x - {}^-3| \overset{\text{want}}{<} 1$$

$$\iff \frac{1}{2} |x - {}^-3| < 1 \iff |x - {}^-3| < 2$$



Now to check endpoints.

$x = -1$

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1 = \infty \quad \Rightarrow \text{dvg.}$$

$x = -5$

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{-4}{4}\right)^n = \sum_{n=1}^{\infty} (-1)^n \quad \text{dvg} \ \underline{\text{osc.}}$$

10. Given radius of conv. of $\sum c_n x^n$ is 16.

So $\sum c_n x^n$ conv. when $|x| < 16$.

Now consider $\sum c_n x^{2n} \stackrel{\text{i.e.}}{=} \sum c_n (x^2)^n$.

It will conv. when $|x^2| < 16$

$$\Updownarrow$$

$$|x|^2 < 16 \iff |x| < 4.$$