HAND IN PART

Spring 2013

Prof. Girardi | Math 142 | [redacted] | 02.21.2013 | Exam 1

MARK BOX		
PROBLEM	POINTS	
0	25	
1	10	
2	10	
3	10	
4	10	
5–11	35	
%	100	

NAME: Key

PIN: 17

INSTRUCTIONS

(1) The MARK BOX above indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(2) You may **not** use an electronic device, a calculator, books, personal notes.

(3) On Problem 0, fill in the blanks. As you were warned, if you do not make at least half of the points on Problem 0, then your score for the entire exam will be whatever you made on Problem 0.

(4) For Problems 1–4, to receive credit you must:

   (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit

   (b) if a line/box is provided, then:
   — show you work BELOW the line/box
   — put your answer on/in the line/box

   (c) if no such line/box is provided, then box your answer.

(5) Problems 5–11 are multiple choice.

- First, indicate (to yourself) your answers directly on the STATEMENT OF MULTIPLE CHOICE PROBLEMS part.
- Once finished with the multiple choice problems, go back to the HAND IN PART and indicate your answers on the table provided.
- Hand in the HAND IN PART. You can take the STATEMENT OF MULTIPLE CHOICE PROBLEMS part home with you (so you can check your answers once the solutions are posted).

(6) During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.

(7) This exam covers (from *Calculus* by Stewart, 6^{th} ed., ET):
7.1–7.5, 7.8, 11.1 .

You were warned about this problem, several times,

0. Fill in the blanks (each worth 1 point).

a. $\int \frac{du}{u}$ = ln $|u| + C$

b. If a is a constant and $a > 0$ but $a \neq 1$, then $\int a^u \, du$ = $\frac{a^u}{\ln a}$ $+C$

c. $\int \cos u \, du$ = sin u $+C$

d. $\int \sec^2 u \, du$ = tan u $+C$

e. $\int \sec u \tan u \, du$ = sec u $+C$

f. $\int \sin u \, du$ = - cos u $+C$

g. $\int \csc^2 u \, du$ = - cot u $+C$

h. $\int \csc u \cot u \, du$ = - csc u $+C$

i. $\int \tan u \, du$ = $-\ln|\cos u|$ or $\ln|\sec u|$ $+C$

j. $\int \cot u \, du$ = $\ln|\sin u|$ or $-\ln|\csc u|$ $+C$

k. $\int \sec u \, du$ = $\ln|\sec u + \tan u|$ or $-\ln|\sec u - \tan u|$ $+C$

l. $\int \csc u \, du$ = $-\ln|\csc u + \cot u|$ or $\ln|\csc u - \cot u|$ $+C$

m. If a is a contant and $a > 0$ then $\int \frac{1}{\sqrt{a^2-u^2}} du$ = $\sin^{-1}(u/a)$ $+C$

n. If a is a contant and $a > 0$ then $\int \frac{1}{a^2+u^2} du$ = $\frac{1}{a}\tan^{-1}(u/a)$ $+C$

o. If a is a contant and $a > 0$ then $\int \frac{1}{u\sqrt{u^2-a^2}} du$ = $1/a \sec^{-1}(|u|/a)$ $+C$

p. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do long division

q. Integration by parts formula: $\int u \, dv$ = $uv - \int v \, du$

r. Trig substitution: (recall that the *integrand* is the function you are integrating) if the integrand involves $a^2 - u^2$, then one makes the substitution u = $a \sin\theta$

s. Trig substitution: if the integrand involves $a^2 + u^2$, then one makes the substitution u = $a \tan\theta$

t. Trig substitution: if the integrand involves $u^2 - a^2$, then one makes the substitution u = $a \sec\theta$

u. trig formula ... your answer should involve trig functions of θ, and not of 2θ: $\sin(2\theta)$ = $2 \sin\theta \cos\theta$.

v. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\cos^2(\theta) = \frac{1}{2}$ $(1 + \cos 2\theta)$.

w. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\sin^2(\theta) = \frac{1}{2}$ $(1 - \cos 2\theta)$.

x. trig formula ... since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween tangent (i.e., tan) and secant (i.e., sec) is $1 + \tan^2\theta = \sec^2\theta$.

y. arctan(-1) = $-\pi/4$ **RADIANS**. (your answer should be an angle)

/2

Problem Source - 100 integrals # 3

1. $\int (\sin x)(\sec x)\,dx = -\ln|\cos x| + C \overset{\text{or}}{=} \ln|\sec x| + C$

$$\int (\sin x)(\sec x)\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}$$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\ln|u| + c = \boxed{-\ln|\cos x| + c}$$

$$\overset{\text{or}}{=} +\ln(|\cos x|)^{-1} + C = \ln\frac{1}{|\cos x|} + C = \boxed{\ln|\sec x| + c}$$

2. $\int x^2 e^x\,dx = x^2 e^x - 2x e^x + 2e^x + C$

Problem Source - Example from class

Examples of Lesson 1

7.3

Ex 1a

$$\int x e^x\,dx = x e^x - \int e^x\,dx = \boxed{x e^x - e^x + C}$$

$u = x \leftrightarrow dv = e^x\,dx$

$du = dx \quad v = e^x$

Ex 1b

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx = x^2 e^x - 2\left(x e^x - \int e^x\,dx\right)$$

$u = x^2 \leftrightarrow dv = e^x\,dx$

$du = 2x\,dx \quad v = e^x$

↓ be careful, comon place for algebra error.

$$= x^2 e^x - 2x e^x \underline{+2} \int e^x\,dx$$

$$= \boxed{x^2 e^x - 2x e^x + 2e^x + C}$$

Note, there is a "d(variable)" on the LHS so there must be a "d(variable)" on the RHS.

Ex 1c How many times would we have to use Parts to find $\int x^{17} e^x\,dx$?

Answer : 17. Why?

Problem Source. Example from our textbook.

3. $\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C$

EXAMPLE 1 Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

SOLUTION Let $x = 3\sin\theta$, where $-\pi/2 \leqslant \theta \leqslant \pi/2$. Then $dx = 3\cos\theta\,d\theta$ and

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

(Note that $\cos\theta \geqslant 0$ because $-\pi/2 \leqslant \theta \leqslant \pi/2$.) Thus the Inverse Substitution Rule gives

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta \\ &= \int (\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C\end{aligned}$$

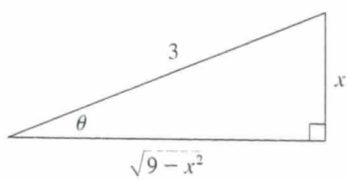


FIGURE 1

$\sin\theta = \frac{x}{3}$

Since this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express $\cot\theta$ in terms of $\sin\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle. Since $\sin\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9-x^2}$, so we can simply read the value of $\cot\theta$ from the figure:

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for $\cot\theta$ is valid even when $\theta < 0$.) Since $\sin\theta = x/3$, we have $\theta = \sin^{-1}(x/3)$ and so

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

4

Problem Source: similar to problems in class.
Remember, divide num. & den. by n (highest power)

4. For the following **SEQUENCES**:

- if the limit exists, find it
- if the limit does not exist, then say that it DNE.

Put your ANSWER IN the box and show your WORK BELOW the box.

4a.

$$\lim_{n\to\infty} \frac{5n^2 + 4\sqrt{n}}{6n^2 + 7n + 1} = \frac{5}{6}$$

$$\frac{5n^2 + 4\sqrt{n}}{6n^2 + 7n + 1} \underset{\div n^2}{=} \frac{5 + \frac{4}{n^{3/2}}}{6 + \frac{7}{n} + \frac{1}{n^2}} \xrightarrow{n\to\infty} \frac{5+0}{6+0+0} = \frac{5}{6}$$

4b.

$$\lim_{n\to\infty} \frac{-5n^8 + 4\sqrt{n}}{6n^3 + 7n^2 + 1} = -\infty$$

$$\frac{-5n^8 + 4n^{1/2}}{6n^3 + 7n^2 + 1} \underset{\div n^8}{=} \frac{-5 + \frac{4}{n^{15/2}}}{\frac{6}{n^5} + \frac{7}{n^6} + \frac{1}{n^8}} \xrightarrow{n\to\infty} \cancel{\frac{-5+0}{0+0+0}} \stackrel{?}{=} \cancel{\pm\infty} = -\infty$$

4c.

$$\lim_{n\to\infty} \frac{5n^3 + 4\sqrt{n}}{6n^8 + 7n^2 + 1} = 0$$

$$\frac{5n^3 + 4n^{1/2}}{6n^8 + 7n^2 + 1} \underset{\div n^8}{=} \frac{\frac{5}{n^5} + \frac{4}{n^{15/2}}}{6 + \frac{7}{n^6} + \frac{1}{n^8}} \xrightarrow{n\to\infty} \frac{0+0}{6+0+0} = 0$$

/5

5. $\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+9} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+9| + C$

$\boxed{\begin{array}{l} u = x^2+9 \\ du = 2x\,dx \end{array}}$

Check $D_x\ \frac{1}{2}\ln|x^2+9| = \frac{1}{2}\ \frac{1}{x^2+9}\cdot 2x = \frac{x}{x^2+9}$ ✓

So $\int_0^1 \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9|\Big|_{x=0}^{x=1} = \frac{1}{2}\ln 10 - \frac{1}{2}\ln 9$

$= \frac{1}{2}[\ln 10 - \ln 9]$

6. $\int \frac{x}{x+9}\,dx = \int 1dx - 9\int \frac{dx}{x+9} = x - 9\ln|x+9| + C$

$\boxed{\begin{array}{l} \frac{x}{x+9} \underset{\uparrow}{=} \frac{x+9}{x+9} - \frac{9}{x+9} \\ \phantom{\frac{x}{x+9}} = 1 - \frac{9}{x+9} \end{array}}$

Long Division (Fake)

Check $D_x[x - 9\ln|x+9|] = 1 - \frac{9}{x+9}$

$= \frac{x+9}{x+9} - \frac{9}{x+9} = \frac{x}{x+9}$. ✓

So $\int_0^4 \frac{x}{x+9}\,dx = [x - 9\ln|x+9|]\Big|_{x=0}^{x=4}$

$= [4 - 9\ln|13|] - [0 - 9\ln|9|]$

$= 4 - 9\ln(13) + 9\ln(9)$

7. $\int_0^1 \frac{dx}{x^p} = \lim_{a\to 0^+}\int_a^1 x^{-p}\,dx = \lim_{a\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_{x=a}^{x=1}$

$= \frac{1}{1-p}\lim_{a\to 0^+}\left[1 - a^{1-p}\right] \underset{\uparrow}{=} \frac{1}{1-p}[1-0] = \frac{1}{1-p}$.

$\boxed{\begin{array}{l} \underline{\text{If}}\ 0 < 1-p \\ \text{i.e. } p < 1 \end{array}}$

So we want $\frac{1}{1-p} = 1.25$

i.e. $\iff \frac{1}{1-p} = \frac{5}{4} \iff 1-p = \frac{4}{5} \iff 1-\frac{4}{5} = p \iff p = \frac{1}{5}$

So $p = \frac{1}{5} = 0.2$

/6

8. $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \, (1-\sin^2 x) \boxed{\cos x \, dx}$

$\boxed{\begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array}}$

$= \int t^2 (1-t^2) \, dt = \int (t^2 - t^4) \, dt$

$= \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$

Check $D_x \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} \right] = \frac{1}{3} \cdot 3 \cdot \sin^2 x \cos x - \frac{1}{5} 5 \sin^4 x \cos x$

$= (\sin^2 x)(\cos x) \left[1 - \sin^2 x \right] = \sin^2 x \cos^3 x.$ ✓

so $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx = \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} \right] \Big|_{x=0}^{x=\pi/2}$

$= \left[\frac{1}{3} - \frac{1}{5} \right] - \left[\frac{0}{3} - \frac{0}{5} \right] = \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15} = \frac{2}{15}.$

/7

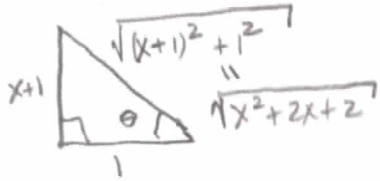
#9

$$\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2} = \int \frac{\sec^2\theta}{\sec^4\theta} \, d\theta$$

$$= \int \frac{1}{\sec^2\theta} \, d\theta$$

$$x^2+2x+2 = (x+1)^2 + 1$$

$$x+1 = \tan\theta \Rightarrow$$



$$dx = \sec^2\theta \, d\theta$$

$$x^2+2x+2 = (x+1)^2+1 = \tan^2\theta + 1 = \sec^2\theta$$

$$= \int \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C = \frac{1}{2}\left[\theta + \frac{1}{2}\cdot 2\cos\theta\sin\theta\right] + C$$

$$= \frac{1}{2}\theta + \cos\theta\sin\theta + C$$

$$= \frac{1}{2}\arctan(x+1) + \frac{1}{\sqrt{x^2+2x+2}} \cdot \frac{x+1}{\sqrt{x^2+2x+2}} + C$$

$$= \frac{1}{2}\arctan(x+1) + \frac{x+1}{x^2+2x+2} + C ,$$

/8

#10 Problem Source: Textbook, § 7.4, # 15

$$\boxed{\int \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = x + \ln|x| - \frac{2}{x} - \ln|x-2| \qquad + C}$$

PFD

Hint: Do we have (Strictly) Bigger Bottoms ?

NO so need to do long division ... but it's easy to "fake" long division on this one (lucky us)

$$\frac{x^3-2x^2-4}{x^3-2x^2} = \frac{x^3-2x^2}{x^3-2x^2} + \frac{-4}{x^3-2x^2} = 1 + \frac{-4}{x^3-2x^2}$$

Find PFD for $\frac{-4}{x^3-2x^2} = \frac{-4}{x^2(x-2)} = \frac{-4}{\underbrace{(x-0)^2}_{(\text{linear term})^2}\underbrace{(x-2)}_{(\text{linear term})^1}}$

$$\frac{-4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$$

$\Rightarrow \boxed{-4 = Ax(x-2) + B(x-2) + Cx^2}$ $\xrightarrow{x=0} -4 = -2B \Rightarrow \boxed{B=2}$

$\xrightarrow{x=2} -4 = C\cdot 2^2 \Rightarrow \boxed{C=-1}$

equate coeff.

$x^2: \; 0 = A + C \xrightarrow{C=-1} \boxed{A = 1}$

$x^1: \; 0 = -2A + B$

$x^0: \; -4 = -2B$

$$\int \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int \left[1 + \frac{1}{x} + \frac{2}{x^2} + \frac{-1}{x-2}\right] dx$$

$$\Downarrow$$

$$\int 2x^{-2}\,dx = \frac{2x^{-1}}{-1} + C$$

So

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \left[4 + \ln 4 - \frac{2}{4} - \ln 2\right] - \left[3 + \ln 3 - \frac{2}{3} - \ln 1\right]$$

$$= \ln 4 - \ln 2 - \ln 3 + 4 - \frac{1}{2} - 3 + \frac{2}{3} = \left(\ln\frac{4}{6}\right) + \frac{7}{6}$$

11. $\lim_{n \to \infty} \left(-\frac{1}{2}\right)^n = 0$

Geometric Series

$r = -\frac{1}{2}$

$|r| = \left|-\frac{1}{2}\right| < 1$

so converges to zero.

$/10$

TABLE FOR YOUR ANSWERS TO MULTIPLE CHOICE PROBLEMS 5 – 11

Instructions.

- Indicate (by circling, boxing, or x-ing) your solution to each problem.
- You may choice up to 3 answers for each problem. The scoring is as follows. For a problem with precisely one answer marked and the answer is correct, 5 points. For a problem with precisely two answers marked, one of which is correct, 3 points. For a problem with precisely three answers marked, one of which is correct, 1 point. All other cases, 0 points.

Your Solutions						
PROBLEM						points
5	5a	5b	5c	5d	5e	
6	6a	6b	6c	6d	6e	
7	7a	7b	7c	7d	7e	
8	8a	8b	8c	8d	8e	
9	9a	9b	9c	9d	9e	
10	10a	10b	10c	10d	10e	
11	11a	11b	11c	11d	11e	