Math 142 . Fall 2012 . Prof. Girardi . Indication of Solutions to 1-14

1.

$$f(x) = \frac{2}{3-x} = \frac{2}{3}\left[\frac{1}{1-\left(\frac{x}{3}\right)}\right] \overset{GS}{=} \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2}{3^{n+1}}x^n .$$

The Geometric Series expansion (GS) is valid $\Leftrightarrow \left|\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3$.

Problem Source : § 11.9 Exercise 5.

2. Start with Geometric Series and take Derivatives as many times as need, Geometric Series is valid when $|x|<1$ so resulting power series expansions will also be valid when $|x|<1$.

$$\text{Geometric Series} \Rightarrow (1-x)^{-1} = \sum_{k=0}^{\infty} x^k \xRightarrow{D_x} (1-x)^{-2} = \sum_{k=1}^{\infty} k x^{k-1}$$

$$\xRightarrow{D_x} 2(1-x)^{-3} = \sum_{k=2}^{\infty} k(k-1)x^{k-2} \xRightarrow{D_x} 2\cdot 3(1-x)^{-4} = \sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3}$$

So

$$(1-x)^{-4} = \sum_{k=3}^{\infty}\frac{k(k-1)(k-2)}{6}x^{k-3} = \sum_{n=0}^{\infty}\frac{(n+3)(n+2)(n+1)}{6}x^n$$

$\boxed{\text{let } k-3=n} \Rightarrow k = n+3$

3

$$\int \tan^{-1}(t^2)\,dt = \int\left[\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(t^2)^{2k-1}}{2k-1}\right]dt$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\left[\int t^{4k-2}\,dt\right] = C + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)}\frac{t^{4k-1}}{4k-1}$$

(note $k=1 \Leftrightarrow k-1=0$ so let $k-1=n$ and so $k=n+1$)

$$= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(4n+3)}t^{4n+3} .$$

Problem Source : Practice Exam 3 # 2.

4

Problem Source § 11.11 Exercise #3

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-1}	$1/2$
1	$-x^{-2}$	$-1/4$
2	$2x^{-3}$	$+1/4$
3	$-6x^{-4}$	$-3/8$

$$P_3(x) = \frac{f^{(0)}(2)}{0!} + \frac{f^{(1)}(2)}{1!}(x-2)^1 + \frac{f^{(2)}(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}\cdot\frac{1}{2}(x-2)^2 - \frac{3}{8}\cdot\frac{1}{2\cdot 3}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

5

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}$
0	x^4-3x^2+1	-1	-1
1	$4x^3-3\cdot 2x$	-2	-2
2	$4\cdot 3x^2-3\cdot 2$	$6=3\cdot 2$	3
3	$4\cdot 3\cdot 2x$	$4\cdot 3\cdot 2$	4
4	$4\cdot 3\cdot 2$	$4\cdot 3\cdot 2$	1
5	0	0	0
$n\geq 5$	0	0	0

Problem Source : § 11.10 Exercise 13.

$$P_\infty(x) = -1-2(x-1)+3(x-1)^2+4(x-1)^3+(x-1)^4$$

6

n	$f^{(n)}(x)$	$f^n(1)$	$\frac{f^n(1)}{n!}$	note
0	x^{-2}	1	1	$1=(-1)^0(0+1)$
1	$-2x^{-3}$	-2	-2	$-2=(-1)^1(1+1)$
2	$+2\cdot 3x^{-4}$	$+3!$	3	$3=(-1)^2(2+1)$
3	$-2\cdot 3\cdot 4x^{-5}$	$-4!$	-4	$-4=(-1)^3(3+1)$
4	$+2\cdot 3\cdot 4\cdot 5x^{-6}$	$5!$	$+5$	$5=(-1)^4(4+1)$
$n>4$			$(-1)^n(n+1)$	checking

Problem Source § 11.10 Exercise # 20.

$$P_\infty(x) = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$$

7 For each $x\in(-1,3)$, there exists $c\in(-1,3)$ so that

$$|R_5(x)| = \left|\frac{f^{(6)}(c)}{6!}(x-0)^6\right| = \frac{1}{6!}e^{-c}|x|^6 \leq \frac{1}{6!}e^{-(-1)}3^6 = \frac{e\,3^6}{6!}$$

Problem Source : Practice Exam # 3, 2012 Fall, # 5.

For # 8-10

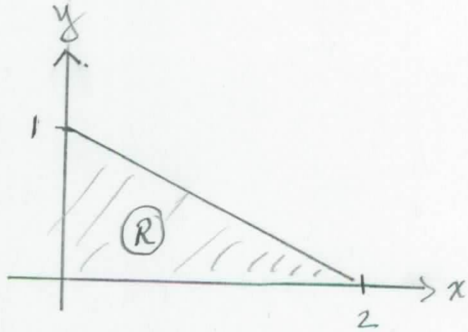
$2y = 2 - x$ ← goes through points: $(0,1)$ & $(2,0)$

$\Updownarrow$

$y = 1 - \frac{x}{2}$

$\Updownarrow$

$x = 2 - 2y$



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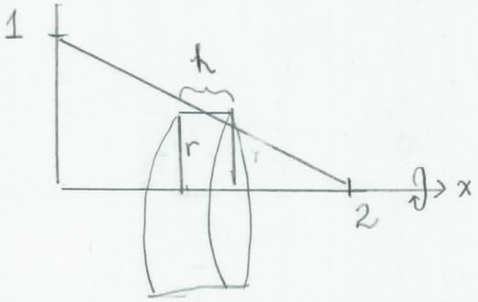


Disk. - all in terms of x.

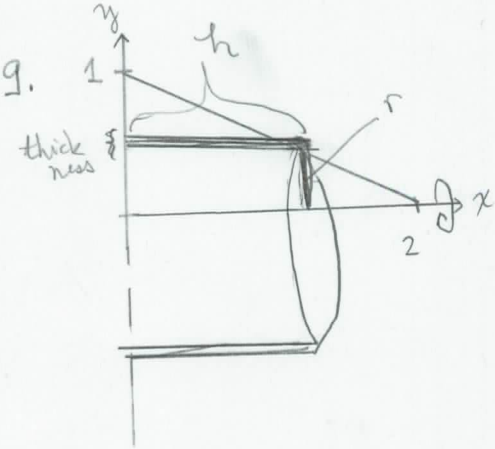
Volume of typical element $\equiv$ disk

$= \pi\,(\text{radius})^2\,(\text{height})$

$= \pi\left(1 - \frac{x}{2}\right)^2 (\Delta x)$

$$\boxed{V = \int_{x=0}^{x=2} \pi\left(1 - \frac{x}{2}\right)^2 dx.}$$

9.



Shell - all in terms of y

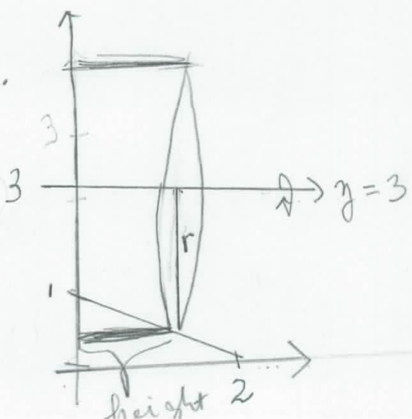
Volume of typical element $\equiv$ shell

$= 2\pi\,(\text{avg. radius})\,(\text{height})\,(\text{thickness})$

$= 2\pi\,(y)\,(2 - 2y)\,\Delta y$

$$\boxed{V = 2\pi \int_{y=0}^{y=1} y(2 - 2y)\,dy}$$

10.



Shell - all in terms of y

Volume of typical element $\equiv$ shell

$= 2\pi\,(\text{avg. radius})\,(\text{height})\,(\text{thickness})$

$= 2\pi\,(3 - y)(2 - 2y)\,\Delta y$

$$\boxed{V = 2\pi \int_{y=0}^{y=1} (3 - y)(2 - 2y)\,dy}$$

Solutions for # 11 - 14

→ problem source : homework problem Ch 7 Review #6
Previous Exam : Fall 2009, Exam 3.

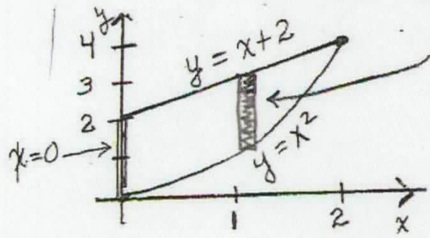
7. Let R be the region in the first quadrant enclosed by

$$y = x^2 \quad \text{and} \quad y = x + 2 \quad \text{and} \quad x = 0 .$$

In each of problems **7a – 7h**, set up an integral or a sum of integrals that express the desired quantity.

- In the space provided **below** each problem, make some *good enough sketch* (does not have to be too fancy) to indicate (i.e., help justify) your thinking/reseasoning behind your solution
- you do not have to do lots of algebra to your integrand
- you do **not have to integrate/evaulate** your integral.

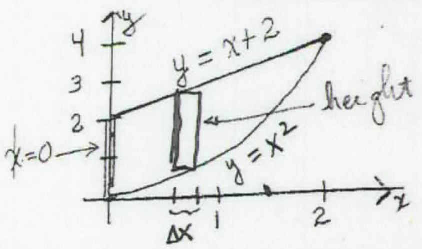
Extra Credit/Hint In the sketch below, draw in a typical rectangle (should it be horizontal or vertical?) that would be used to express the area of R as precisely 1 integral (and not 2 integrals).



7a. The area A of the region R by integrating with respect to x.

#11

$$A = \int_{x=0}^{x=2} \left[x + 2 - x^2 \right] dx$$

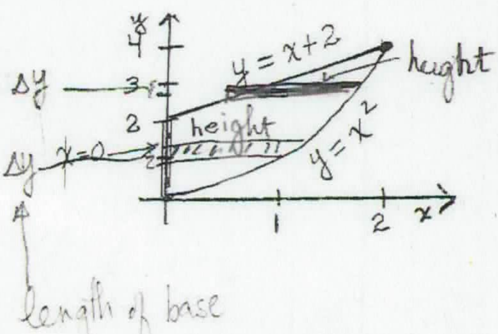


Area Typical rectangle
$= (\text{height})(\text{length of base})$
$= \left[(x+2) - x^2 \right] \Delta x .$

7a. The area A of the region R by integrating with respect to y.

#13

$$A = \int_{y=0}^{y=2} \sqrt{y}\, dy + \int_{y=2}^{y=4} \left[\sqrt{y} - (y-2) \right] dy$$



Area Typical rectangle
$= (\text{height}) \underbrace{(\text{length of base})}_{\Delta y}$

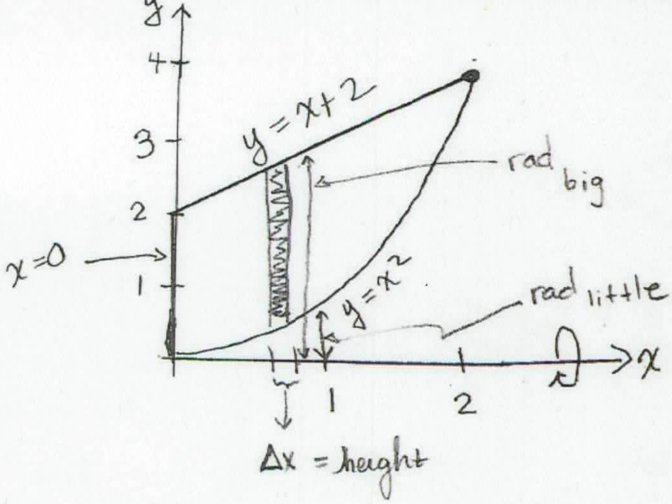
For $0 \le y \le 2$: height $= \sqrt{y} - 0$

For $2 \le y \le 4$: height $= (\sqrt{y}) - (y-2)$

#12

7b. The volume V of the solid obtained by revolving the region R about the x-axis by integrating with respect to x.

$$V = \pi \int_{x=0}^{x=2} \left[(x+2)^2 - (x^2)^2 \right] dx$$



Washer Method

b/c revolving abt x || integrating wrt y

& there's a hole

Volume of Typical Washer

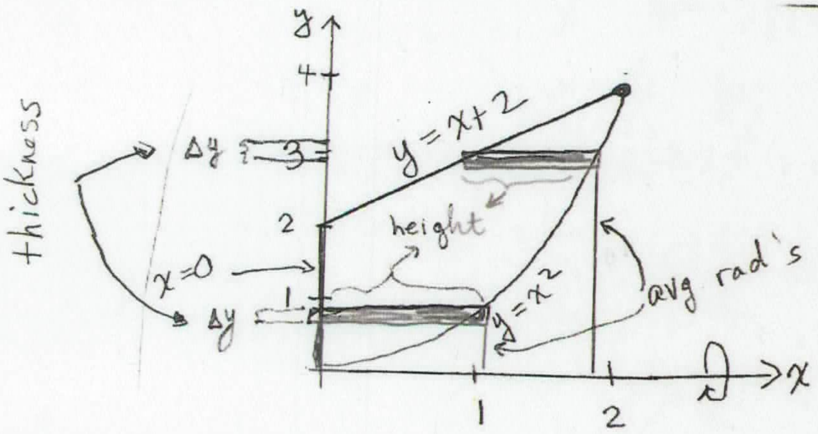
$= \pi \left[(\text{rad}_{\text{big}})^2 - (\text{rad}_{\text{little}})^2 \right] (\text{height})$

$= \pi \left[(x+2)^2 - (x^2)^2 \right] \Delta x$

#14

7c. The volume V of the solid obtained by revolving the region R about the x-axis by integrating with respect to y.

$$\boxed{V = 2\pi \int_{y=0}^{y=2} y\sqrt{y}\,dy + 2\pi \int_{y=2}^{y=4} y\left[(\sqrt{y}) - (y-2)\right] dy}$$



Shell Method
b/c revolving abt x
& integrating wrt y

Volume of Typical Shell

$$= 2\pi\,\underbrace{(\text{avg. radius})}_{y}\,(\text{height})\,\underbrace{(\text{thickness})}_{=\,\Delta y}$$

height: see your last part of 7a.