MARK BOX		
PROBLEM	POINTS	
1 a–k	35	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
%	100	

NAME: Key

PIN: ____________________

INSTRUCTIONS:

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) If you do not make at least 17.5 out of the 35 points on Problem 1, then your score for the entire exam will be whatever you made on Problem 1.
(6) This exam covers (from *Calculus* by Stewart, 6th ed., ET): 11.2–11.8 .

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the above Instructions.

Signature : ____________________

1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

1a. n^{th}-term test for an arbitrary series $\sum a_n$.

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ divergent.

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ < 1
- diverges if and only if $|r|$ ≥ 1

1c. p-series where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p > 1
- diverges if and only if p ≤ 1

1d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

- $a_n = f(n)$ for each $n \in \mathbb{N}$
- f is a continuous function
- f is a positive function
- f is a decreasing function .

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

1e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$. (Fill in the blanks with a_n and/or b_n.)

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converge, then $\sum a_n$ converge.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverge, then $\sum a_n$ diverge.

1f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

1g. Ratio and Root Tests for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$.

Let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.

- If ρ < 1 then $\sum a_n$ converges absolutely.
- If ρ > 1 then $\sum a_n$ diverges.
- If ρ $= 1$ then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

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1i. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

1j. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=17}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

Does $\sum |a_n|$ converge? Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$ Does $\lim_{n\to\infty} |a_n| = 0$?

if NO $\Rightarrow$ $\sum a_n$ is divergent

if YES $\Downarrow$ (from the first box): $\sum a_n$ is absolutely convergent

if YES $\Downarrow$ (from the second box): Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is conditionally convergent

1k. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

T F If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

T F If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges

T F If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.

T F If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.

T F If $S_N = \sum_{n=1}^{N} r^n$, then $S_N = \dfrac{r - r^{N+1}}{1-r}$, for when $r \neq 1$ since we don't like to divide by zero.

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2. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

Check for absolute convergence.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

p-series
$p = 2 > 1$
So it's convergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ is absolutely convergent}}$$

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3. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{4n^7 - n^6 + 21}{11n^8 - n + 17}$$

- [ ] absolutely convergent
- [ ] conditionally convergent
- [x] divergent

You may use, without showing, that for $n \geq 1$

$$a_n := \frac{4n^7 - n^6 + 21}{11n^8 - n + 17} > 0 .$$

Hint. What would be easier: CT or LCT?

$$\frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \overset{n \text{ big}}{\approx} \frac{1}{n} = b_n$$

LCT

$$\frac{a_n}{b_n} = \frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \cdot \frac{n}{1} = \frac{4n^8 - n^7 + 21n}{11n^8 - n^2 + 17n} \xrightarrow{n \to \infty} \frac{4}{11} = L$$

$0 < L < \infty$

$0 < 4/11 < \infty$ ← by LCT a_n + b_n do the same thing

$b_n = \frac{1}{n}$ $p = 1 \leq 1$, so diverges

so by LCT

$\Sigma\, a_n$ diverges.

nice

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Comments on # 3.

The calculations on the following page shows that if $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ then

$$\frac{4}{11n} \leq \frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \iff n = 1$$

but

$$\frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \leq \frac{4}{11n} \iff n \geq 2.$$

Note

$\boxed{n \geq 1}$

$$\frac{4}{11n} \overset{*}{<} \frac{4n^7 - n^6 + 21}{11n^8 - n + 17}$$ both positive for $n \geq 1$

$$\iff 4(11n^8 - n + 17) < 11n(4n^7 - n^6 + 21)$$

$$\iff 44n^8 - 4n + 68 < 44n^8 - 11n^7 + 231n$$

$$\iff 0 < -11n^7 + 235n - 68.$$

Let $f(x) = -11x^7 + 235x - 68$

$f'(x) = -77x^6 + 235$

$f'(x_0) = 0 \iff x_0 = \sqrt[6]{\frac{235}{77}} \approx 1.2$

$$\underset{x_0}{\overset{f'>0 \quad | \quad f'<0}{\overline{\qquad\qquad\qquad\qquad}}}$$

$f(x_0) \approx 175.$

$\lim_{x \to 0} f(x) = -\infty$

$f(1) = 156$

$f(2) = -1006$

$.\overline{18} = \frac{4}{11(2)}$ but $\frac{4(2)^7 - 2^6 + 21}{11(2)^8 - 2 + 17} = \frac{469}{2831} \cong .166$

4. Let

$$a_n = \frac{n!}{(2n-1)!}$$

4a. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{n+1}{4n^2+2n}}$$ ✓

4b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n-1)!}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

$$a_n = \frac{n!}{(2n-1)!} \qquad a_{n+1} = \frac{(n+1)!}{(2(n+1)-1)!} = \frac{(n+1)!}{(2n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2n+1)!} \cdot \frac{(2n-1)!}{n!} = \frac{(n+1)!}{n!} \cdot \frac{(2n-1)!}{(2n+1)!} = \frac{\cancel{n!}\,(n+1)}{\cancel{n!}} \cdot \frac{\cancel{(2n-1)!}}{\cancel{(2n-1)!}\,(2n)(2n+1)}$$

$$= \frac{n+1}{(2n)(2n+1)} = \frac{n+1}{4n^2+2n}$$

Let's check for absolute conv. using the Ratio Test.

$$\lim_{n\to\infty} \frac{n+1}{4n^2+2n} = 0 < 1$$ then abs. conv. by the Ratio Test

nice.

details

$$\lim_{n\to\infty} \frac{n+1}{4n^2+2n} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{4+\frac{2}{n}} = \frac{0+0}{4+0} = 0$$

$\div$ by n (highest power) $= n^2$

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5. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{3n^2+8n-1}$$

- [ ] absolutely convergent
- [ ] conditionally convergent
- [x] divergent

First note $\frac{n^2+1}{3n^2+8n-1} > 0$ for $n \in \mathbb{N}$.

Check abs. conv.

$$\lim_{n\to\infty} \left| (-1)^n \frac{n^2+1}{3n^2+8n-1} \right| = \lim_{n\to\infty} \frac{n^2+1}{3n^2+8n-1} \overset{(*)}{=} \frac{1}{3} \neq 0.$$

So by n^{th} term test for divg., $\sum \frac{n^2+1}{3n^2+8n-1}$ dvg.

So $\sum (-1)^n \frac{n^2+1}{3n^2+8n-1}$ is not abs. conv.

Well (*) tells us what to do next!

By the n^{th}-term test for dvg, $\sum (-1)^n \frac{n^2+1}{3n^2+8n-1}$ diverges since

$\lim_{n\to\infty} (-1)^n \frac{n^2+1}{3n^2+8n-1}$ DNE b/c it osc. (see (*)). We are done.

Comments on #5 due to common mistakes on exam.

Note $\sum (-1)^n \frac{n^2+1}{3n^2+8n-1}$ is an alternating series $\sum (-1)^n u_n$ with

$$u_n = \frac{n^2+1}{3n^2+8n-1} > 0.$$

continued →

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In order to apply the AST, both conditions for the test, namely:

① $u_n > u_{n+1}$ for all n sufficiently large, say for all $n \geq N$,

② $\lim_{n \to \infty} u_n = 0$,

N is any (big) number

"doesn't matter where we start."

must be satisfied. If both ① and ② are not satisfied, then the AST "fails" (ie the AST can not be applied) and so you need to use another test.

Let's check ① : $u_n > u_{n+1}$?

① is not obvious so let's use the first derivative test.

Let $f(x) = \dfrac{x^2+1}{3x^2+8x-1}$ for $x > 1$. Some basic calculus gives

$f'(x) = \dfrac{8(x^2-x-1)}{[3x^2+8x-1]^2}$. Note $y = x^2-x-1$ is a ccu parabola with roots at $\dfrac{1 \pm \sqrt{5}}{2}$. So

$$f'(x) > 0 \quad \text{for } x > \frac{1+\sqrt{5}}{2}$$

So

$$u_n < u_{n+1} \quad \text{for } n > \frac{1+\sqrt{5}}{2}.$$

So ① does not hold; thus, the AST cannot be applied.

But if $0 < u_n < u_{n+1}$ for n sufficiently big N, then you can show that $\lim_{n \to \infty} (-1)^n u_n \neq 0$ (think why this is so) so you know to try the n^{th} term test for dvg to show $\Sigma(-1)^n u_n$ dvg.

2> Now let's see what happens if you checked condition ②.

Let's check condition ② : $\lim_{n\to\infty} u_n = 0$?

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2+1}{3n^2+8n-1} \underset{\uparrow}{=} \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{3+\frac{8}{n}-\frac{1}{n^2}} = \frac{1-0}{3+0+0} = \frac{1}{3} \neq 0.$$

$\div$ by $n^{\text{highest power}}$, ie, $\div n^2$

So ② does not hold; thus AST cannot be applied.

But this tells us we can apply the n^{th} term test for dvg to $\sum (-1)^n u_n$:

$\lim_{n\to\infty} (-1)^n \dfrac{n^2+1}{3n^2+8n-1}$ DNE b/c it oscilates so

by n^{th} term test for dvg, $\sum (-1)^n \dfrac{n^2+1}{3n^2+8n-1}$ dvg.

What have you learned from this?

If both conditions ① & ② of AST are not satisfied, then the AST cannot be applied; but, then we should use the n^{th} term test for dvg to $\sum (-1)^n u_n$.

Homework problem § 11.7 # 19

6. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$$

- [ ] absolutely convergent
- [x] conditionally convergent
- [ ] divergent

19. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/(2\sqrt{n})} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

Now need to consider

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}} \quad \underline{\underline{\text{and}}} \text{ Let } a_n = \frac{\ln n}{n^{1/2}}$$

WAY # 1

For $n \geq 3$

$$\frac{1}{n^{1/2}} \leq \frac{\ln n}{n^{1/2}}$$

$$\sum_{n=3}^{\infty} \frac{1}{n^{1/2}} = \infty \quad (\text{p-series, } p = \tfrac{1}{2} < 1) \text{ so}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}} = \infty \text{ by CT.}$$

WAY # 2

Limit Comparison Test (LCT)	Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.	
	If $0 < L < \infty$, then [$\sum a_n$ conv. $\iff$ $\sum b_n$ conv.]	(you DO need to memorize this one)
	If $L = 0$, then [$\sum b_n$ conv. $\implies$ $\sum a_n$ conv.]	(you do not have to memorize this one)
	⊛ → If $L = \infty$, then [$\sum b_n$ divg. $\implies$ $\sum a_n$ divg.]	(you do not have to memorize this one)

Let $b_n = \frac{1}{\sqrt{n}}$. Note $\sum b_n$ dvg b/c p-series, $p = 1/2 < 1$.

$$\frac{a_n}{b_n} = \frac{\ln n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \ln n \xrightarrow{n\to\infty} \infty.$$

So $\sum a_n = \sum \frac{\ln n}{\sqrt{n}}$ dvg.

If you use WAY # 2, you must state the version of the LCT you are using since it is not one to memorize & listed on question 1 of this exam.

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7. Geometric Series. (On this page, you should do basic algebra but you do NOT have to do any fancy arithmetic (eg, just making up numbers, you can leave $(\frac{17}{18})^{171}$ as just that.) Let, for $N \geq 51$,

$$s_N = \sum_{n=51}^{N} 2\frac{3^{n+1}}{5^n}.$$

7a. Do some algebra to write s_N as $\sum_{n=51}^{N} c\, r^n$ for an appropriate constant c and ratio r.

$$\boxed{s_N = \sum_{n=51}^{N} 6\left(\frac{3}{5}\right)^n}$$

$$\frac{2 \cdot 3^1 \cdot 3^n}{5^n}$$

7b. Using the method from class (rather than some formula), find an expression for s_N in closed form (i.e. without a summation $\sum$ sign nor some dots . . . .).

$$\boxed{s_N = 15\left[\left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1}\right]}$$

$$S_N = 6\left[\left(\frac{3}{5}\right)^{51} + \left(\frac{3}{5}\right)^{52} + \left(\frac{3}{5}\right)^{53} + \cdots + \left(\frac{3}{5}\right)^{N}\right]$$

$$- \quad \frac{3}{5} S_N = 6\left[\quad + \left(\frac{3}{5}\right)^{52} + \left(\frac{3}{5}\right)^{53} + \cdots \left(\frac{3}{5}\right)^{N} + \left(\frac{3}{5}\right)^{N+1}\right]$$

$$\frac{2}{5} S_N = 6\left[\left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1}\right]$$

7c. Does $\sum_{n=51}^{\infty} 2\frac{3^{n+1}}{5^n}$ converge or diverge? If it converges, find its sum. Justify your answer.

$$\boxed{\sum_{n=51}^{\infty} 2\frac{3^{n+1}}{5^n} \quad 15\left(\frac{3}{5}\right)^{51}}$$

$$\lim_{n \to \infty} 15\left[\left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1}\right] \qquad \lim_{n \to \infty} \left(\frac{3}{5}\right)^{N+1} = 0$$

since $\left|\frac{3}{5}\right| < 1$

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