1. $\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+9| + C$ ✓

$\boxed{\begin{array}{l} u = x^2+9 \\ du = 2x\,dx \end{array}}$

$\int_0^1 \frac{x\,dx}{x^2+9} = \frac{1}{2}\ln|x^2+9|\Big|_0^1 = \frac{1}{2}\left[\ln 10 - \ln 9\right] = \frac{1}{2}\ln\left(\frac{10}{9}\right)$ ✓

2. $\int \frac{x-1}{x^2+2x}\,dx = \int \left[-\frac{1}{2}\,\frac{1}{x} + \frac{3}{2}\,\frac{1}{x+2}\right]dx = -\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x+2| + C$

or

$= \frac{1}{2}\left[3\ln|x+2| - \ln|x|\right] + C$ ✓

$\frac{x-1}{x^2+2x} = \frac{x-1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} = \frac{A(x+2) + Bx}{x(x+2)}$

$x - 1 = (A+B)x + 2A$

$x^1 : \; 1 = A + B$

$x^0 : \; -1 = 2A \Rightarrow A = \frac{-1}{2}$

$\Rightarrow B = \frac{3}{2}$

3. $\int \frac{\sin 2x}{1+\cos^4 x}\,dx = \int \frac{2\sin x \cos x}{1+\cos^4 x}\,dx = -\int \frac{2u\,du}{1+(u^2)^2} = -\int \frac{ds}{1+s^2}$

$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}}$

$\boxed{\begin{array}{l} s = u^2 \\ ds = 2u\,du \end{array}}$

$= -\arctan s + C = -\arctan u^2 + C$

$= -\arctan(\cos^2 x) + C$

4. $\displaystyle\int \frac{x^2}{x+2}\,dx = \int \left(x - 2 + \frac{4}{x+2}\right)dx = \frac{x^2}{2} - 2x + 4\ln|x+2| + C$ 2

$$\begin{array}{r} x - 2 \\ x+2 \;\overline{)\; x^2 + 0x + 0} \\ \underline{x^2 + 2x} \\ -2x + 0 \\ \underline{-2x - 4} \\ 4 \end{array}$$

#5 - continued

$$\int_0^{\pi/6} x \sin x \cos x \, dx = \left[\frac{\sin(2x)}{8} - \frac{x\cos(2x)}{4}\right]\Bigg|_{x=0}^{x=\pi/6}$$

$$= \left(\frac{\sin\frac{\pi}{3}}{8} - \frac{\pi}{6}\,\frac{\cos\pi/3}{4}\right) - 0 = \frac{1}{8}\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{24}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{16} - \frac{\pi}{48}$$

5. $\displaystyle\int x \sin x \cos x \, dx = \frac{1}{2}\int x \, 2\sin x \cos x \, dx = \frac{1}{2}\int x \sin(2x)\,dx$

$u = x \qquad dv = \sin(2x)\,dx$

$du = dx \qquad v = -\frac{1}{2}\cos(2x)$

$$= \frac{1}{2}\left[-\frac{x}{2}\cos(2x) - -\frac{1}{2}\int \cos 2x \, dx\right]$$

$$= -\frac{x}{4}\cos 2x + \frac{1}{4}\int \cos 2x \, dx$$

$$= -\frac{x}{4}\cos 2x + \frac{1}{4}\,\frac{\sin 2x}{2} + C = \frac{\sin(2x)}{8} - \frac{x\cos(2x)}{4} + C$$

$$= \frac{2\cos x \sin x}{8} - \frac{x(\cos^2 x - \sin^2 x)}{4} + C$$

$$= \frac{\cos x \sin x}{4} - \frac{x\cos^2 x}{4} + \frac{x\sin^2 x}{4} = \frac{\cos x \sin x}{4} - \frac{x(1-\sin^2 x)}{4} + \frac{x\sin^2 x}{4} + C$$

$$= \frac{\cos x \sin x}{4} - \frac{x}{4} + \frac{x\sin^2 x}{2} + C$$

or

$u = x \qquad dv = \cos x \sin x \, dx$

$du = dx \qquad v = \frac{\sin^2 x}{2}$

$$= \frac{x\sin^2 x}{2} - \frac{1}{2}\int \sin^2 x \, dx$$

$$= \frac{x\sin^2 x}{2} - \frac{1}{2}\,\frac{1}{2}\int (1 - \cos 2x)\,dx$$

$$= \frac{x\sin^2 x}{2} - \frac{1}{4}\int dx + \frac{1}{4}\,\frac{1}{2}\int \cos(2x)\cdot 2\,dx = \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{8}\sin(2x)$$

$$= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{8}\cdot 2\sin x \cos x + C = \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{\sin x \cos x}{4}$$

6. $\int \sin^2 x \cos^3 x \, dx = \int (\sin^2 x)(1-\sin^2 x) \cos x \, dx$

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}}$

$= \int u^2(1-u^2)\, du = \int (u^2 - u^4)\, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$ ✓

$\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} \Big|_{x=0}^{x=\pi/2} = \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15} = \frac{2}{15}$ ✓

7. $\int \frac{x^3}{\sqrt{x^2+25}}\, dx = \int \frac{5^3 \tan^3\theta \cdot 5\sec^2\theta \, d\theta}{5\sec\theta} = 5^3 \int \tan^3\theta \sec\theta \, d\theta$

$= 5^3 \int (\sec^2\theta - 1) \boxed{\tan\theta \sec\theta \, d\theta}$

$= 5^3 \int (u^2-1)\, du = 5^3 \left[\frac{u^3}{3} - u\right] + C$

$= 5^3 \left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C.$

$= \frac{5^3}{3} \frac{(x^2+25)^{3/2}}{5^3} - 5^3 \frac{(x^2+25)^{1/2}}{5} + C$

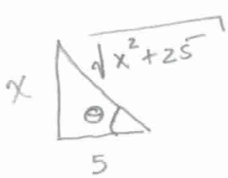
$= \frac{(x^2+25)^{3/2}}{3} - 25(x^2+25)^{1/2} + C$ (✓)

$\boxed{\begin{array}{l} x = 5\tan\theta \\ dx = 5\sec^2\theta \, d\theta \\ \sqrt{x^2+25} = \sqrt{25(\tan^2\theta + 1)} \\ \qquad = 5\sec\theta \end{array}}$

$\boxed{\begin{array}{c} c^2 + s^2 = 1 \\ \Downarrow \\ 1 + \tan^2\theta = \sec^2\theta \end{array}}$

$\boxed{\begin{array}{l} u = \sec\theta \\ du = \sec\theta\tan\theta \, d\theta \end{array}}$

$x = 5\tan\theta \Rightarrow \tan\theta = \frac{x}{5}$



$\Rightarrow \sec\theta = \frac{(x^2+25)^{1/2}}{5}$

8. $\int \frac{1+5e^x}{1-e^x}\, dx = \int \left[\frac{1-e^x}{1-e^x} + \frac{6e^x}{1-e^x}\right] dx = \int 1\, dx + 6\int \frac{e^x}{1-e^x}\, dx$

$\boxed{\begin{array}{l} u = 1 - e^x \\ du = -e^x \, dx \end{array}}$

$= x + -6 \int \frac{du}{u} = x - 6\ln|1-e^x| + C$ ✓

9. $\boxed{x>1}$

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$$

$$\int_1^\infty \frac{dx}{x\ln x} = \int_1^2 \frac{dx}{x\ln x} + \int_2^3 \frac{dx}{x\ln x} + \int_3^\infty \frac{dx}{x\ln x}$$

$$= \left[\lim_{a\to 1^+} \int_a^2 \frac{dx}{x\ln x}\right] + \left[\int_2^3 \frac{dx}{x\ln x}\right] + \lim_{b\to\infty}\left[\int_3^b \frac{dx}{x\ln x}\right]$$

$\boxed{\ln 1 = 0}$

$$= \left[\lim_{a\to 1^+} \ln|\ln 2| - \ln|\ln a|\right] + \left[\ln|\ln 3| - \ln|\ln 2|\right] + \lim_{b\to\infty}\left[\ln|\ln b| - \ln|\ln 3|\right]$$

$$= \ln|\ln 2| - \left[\lim_{a\to 1^+} \ln|\ln a|\right] + \ln|\ln 3| - \ln|\ln 2| + \left[\lim_{b\to\infty} \ln|\ln b|\right] - \ln|\ln 3|$$

$$= \underbrace{\lim_{b\to\infty} \ln|\ln b|}_{\infty} - \lim_{a\to 1^+} \ln|\underbrace{\ln a}_{\to 0}|$$

$$\infty \quad - \quad -\infty \quad = \infty + \infty = \infty.$$

10. For $p \neq 1$, we have the following.

$$\int_0^1 x^{-p}\,dx = \frac{x^{1-p}}{1-p}\Big|_0^1 = \frac{1}{1-p}\left[1^{1-p} - 0^{1-p}\right] = \frac{1}{1-p} \overset{\text{want}}{=\!=} 1.25$$

$\boxed{\text{need } 1-p>0, \text{ i.e. } 1>p}$

$$1.25 = 1\tfrac{1}{4} = \frac{5}{4} \overset{\text{want}}{=\!=} \frac{1}{1-p}.$$

$$\Updownarrow$$

$$1-p = \frac{4}{5} \iff p = 1 - \frac{4}{5} = \frac{1}{5} = 0.2$$

11. $\int \frac{dx}{4x^2+4x+5} = \int \frac{dx}{(2x+1)^2+2^2} = \int \frac{\sec^2\theta\, d\theta}{2^2 \sec^2\theta} = \frac{1}{4}\int d\theta$

$2x+1 = 2\tan\theta$ $\qquad = \frac{1}{4}\theta + C$

$2\,dx = 2\sec^2\theta\, d\theta$ $\qquad = \frac{1}{4}\tan^{-1}\left[\frac{2x+1}{2}\right] + C$ ✓

$(2x+1)^2 + 2^2 = 2^2\tan^2\theta + 2^2$

$= 2^2\sec^2\theta$

Note $4x^2+4x+5 = (2x+1)^2 + 2^2 \geq 0$ so ....

$$\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5} = \int_{-\infty}^{0} \frac{dx}{4x^2+4x+5} + \int_{0}^{\infty} \frac{dx}{4x^2+4x+5}$$

$$= \lim_{a\to-\infty}\left[\int_a^0 \frac{dx}{4x^2+4x+5}\right] + \lim_{b\to\infty}\left[\int_0^b \frac{dx}{4x^2+4x+5}\right]$$

$$= \lim_{a\to-\infty}\left[\frac{1}{4}\tan^{-1}\frac{1}{2} - \frac{1}{4}\tan^{-1}\left(\frac{2a+1}{2}\right)\right] +$$

$$\lim_{b\to\infty}\left[\frac{1}{4}\tan^{-1}\left(\frac{2b+1}{2}\right) - \frac{1}{4}\tan^{-1}\frac{1}{2}\right]$$

$$= \left[\lim_{b\to\infty}\frac{1}{4}\tan^{-1}\left(\frac{2b+1}{2}\right)\right] + \left[\lim_{a\to-\infty}\frac{1}{4}\tan^{-1}\left(\frac{2a+1}{2}\right)\right]$$

$$= \frac{1}{4}\frac{\pi}{2} + \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{4}.$$ ✓

12. $\int e^t \sqrt{25 - e^{2t}}\, dt = \int \sqrt{5^2 - u^2}\, du$

$u = e^t$
$du = e^t\, dt$

$= \int 5\cos\theta \cdot 5\cos\theta\, d\theta$

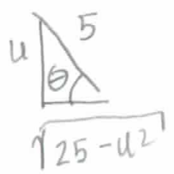
$= 5^2 \int \cos^2\theta\, d\theta = \frac{5^2}{2}\int (1 + \cos 2\theta)\, d\theta$

$u = 5 \sin\theta$
$du = 5\cos\theta\, d\theta$
$\sqrt{5^2 - u^2} = \sqrt{5^2(1 - \sin^2\theta)}$
$= 5\cos\theta$

$= \frac{5^2}{2}\int d\theta + \frac{5^2}{2}\,\frac{1}{2}\int (\cos(2\theta))(2)\, d\theta$

$= \frac{5^2}{2}\theta + \frac{5^2}{4}\sin(2\theta) + C$

$\sin\theta = \frac{u}{5}$



$\Rightarrow \cos\theta = \frac{\sqrt{25 - u^2}}{5}$

$= \frac{5^2}{2}\sin^{-1}\left(\frac{u}{5}\right) + \frac{5^2}{4} \cdot 2\sin\theta\cos\theta + C$

$= \frac{25}{2}\sin^{-1}\left(\frac{u}{5}\right) + \frac{5^2}{2}\,\frac{u}{5}\cdot\frac{\sqrt{25 - u^2}}{5} + C$

$= \frac{25}{2}\sin^{-1}\left(\frac{e^t}{5}\right) + \frac{e^t}{2}\sqrt{25 - e^{2t}} + C$ ✓

13. $\int \frac{\ln(\tan x)}{(\sin x)(\cos x)}\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\left[\ln|\tan x|\right]^2 + C$ ✓

$u = \ln(\tan x)$
$du = \frac{1}{\tan x} \cdot \sec^2 x\, dx$
$= \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}\, dx$
$= \frac{dx}{\sin x \cos x}$

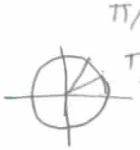
$\int_{\pi/6}^{\pi/3} \frac{\ln(\tan x)\, dx}{\sin x \cos x} = \frac{1}{2}\left[(\ln|\tan x|)^2\Big|_{\pi/6}^{\pi/3}\right]$

$= \frac{1}{2}\left[(\ln|\tan \pi/3|)^2 - (\ln|\tan \pi/6|)^2\right]$

$= \frac{1}{2}\left[(\ln(3^{1/2}))^2 - (\ln(3^{-1/2}))^2\right] =$

$= \frac{1}{2}\left[\left(\frac{1}{2}\ln 3\right)^2 - \left(-\frac{1}{2}\ln 3\right)^2\right]$

$= 0$ ✓



$\pi/3\ (1/2, \sqrt{3}/2) : \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \sqrt{3} = 3^{1/2}$

$\pi/6\ (\frac{\sqrt{3}}{2}, \frac{1}{2}) : \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = 1/\sqrt{3} = 3^{-1/2}$

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14.

$$\int e^{-2x}\,dx = \frac{-1}{2}\int e^{-2x}\,(-2\,dx) = -\frac{1}{2}e^{-2x} + C$$

$$\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b e^{-2x}\,dx = \lim_{b\to\infty} -\frac{1}{2}e^{-2x}\Big|_{x=0}^{x=b}$$

$$= -\frac{1}{2}\left[\lim_{b\to\infty}\left(e^{-2b} - e^0\right)\right] = \frac{-1}{2}\left[0-1\right] = \frac{1}{2}$$

15. 

16. Algebra Time:

$$\frac{1}{-e^{-x}+e^x} = \frac{1}{e^x - \frac{1}{e^x}} = \frac{1}{\frac{e^{2x}-1}{e^x}} = \frac{e^x}{(e^x)^2-1}.$$

So for calculus

$$\int \frac{dx}{-e^{-x}+e^x} = \int \frac{e^x}{(e^x)^2-1}\,dx = \int \frac{du}{u^2-1} = -\frac{1}{2}\int\frac{du}{u+1} + \frac{1}{2}\int\frac{du}{u-1}$$

$u = e^x$

$du = e^x\,dx$

$$= -\frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| + C$$

$$= \frac{1}{2}\left[\ln\left|\frac{u-1}{u+1}\right|\right] + C = \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C.$$

$$\frac{1}{u^2-1} = \frac{1}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} = \frac{A(u-1)+B(u+1)}{(u+1)(u-1)}$$

$$1 = A(u-1) + B(u+1)$$

$$1 = (A+B)u + (-A+B)$$

$$0 = A+B \Rightarrow -A = B$$

$$1 = -A+B$$

$$\Rightarrow 1 = 2B$$

$B = 1/2$ and $A = -1/2$

17. $\int x \ln(x+1)\,dx = \frac{x^2 \ln(x+1)}{2} - \frac{1}{2}\int \frac{x^2}{x+1}\,dx$

$u = \ln(x+1) \qquad dv = x\,dx$

$du = \frac{dx}{x+1} \qquad v = \frac{x^2}{2}$

$= \frac{x^2 \ln(x+1)}{2} - \frac{1}{2}\int \left[(x-1) + \frac{1}{x+1}\right] dx$

$$\begin{array}{r} x \;\; -1 \\ x+1 \;\big)\; x^2 + 0x + 0 \\ \underline{x^2 + 1x} \\ -1x \;\; -0 \\ \underline{-x \;\; -1} \\ +1 \end{array}$$

$= \frac{x^2 \ln(x+1)}{2} - \frac{1}{2}\left[\frac{x^2}{2} - x + \ln|x+1|\right] + C$

$= \frac{x^2 \ln|x+1|}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{\ln|x+1|}{2} + C$ ✓

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