

MARK BOX		
PROBLEM	POINTS	
1	3	
2: a - i	27	
TOTAL	30	

NAME: Key
class PIN: 17

INSTRUCTIONS:

- (1) To receive credit you must:
 - (a) **work in a logical fashion, show all your work, indicate your reasoning;**
no credit will be given for an answer that just appears;
such explanations help with partial credit
 - (b) if a line/box is provided, then:
 - show your work BELOW the line/box
 - put your answer on/in the line/box
 - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
- (3) You may use your notes and the textbook. You cannot use eachother (i.e., you have to take this part solo, without the help of someone else).
- (4) This exam covers (from *Calculus* by Stewart 6th ed,ET): § 11.9, 11.10, 11.11 .

Problem Inspiration: just like the homework.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Furthermore, I have not only read but will also follow the above Instructions.

I hereby verify that I did NOT receive help from other people on this take-home exam problem.

Signature : 

**Due Tuesday Novemeber 22, 2011 at the start of class (12:30pm).
No Exceptions!**

Taylor/Maclaurin Polynomials and Series

1. Using the known Taylor Series

$$\ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n, \quad t \in (-1, 1], \text{ i.e., } -1 < t \leq 1$$

(as from the handout **Commonly Used Taylor Series**) and methods from Section 11.9, find a power series expansion (in CLOSED form) for

$$y = \ln(x-4) \text{ about the center of } x_0 = 5.$$

Hint: $\ln(x-4) = \ln[1+(x-5)]$. Also, say when this power series expansion is valid.

$\ln(x-4) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-5)^n$

$\frac{(-1)^{n-1}}{n}$
 $(x-5)^n$
which is valid for
4
 $< x \leq$
6

just let $t = x-5$

Good for

$$\begin{array}{r}
 -1 < x-5 \leq 1 \\
 +5 \quad +5 \quad +5 \\
 \hline
 4 < x \leq 6
 \end{array}$$

}

So $\ln(x-4) = 1(x-5)^1 - \frac{1}{2}(x-5)^2 + \frac{1}{3}(x-5)^3 - \frac{1}{4}(x-5)^4 + /- \dots$

2. Do parts (a) - (i) for the following problem.

$$f(x) = \ln(x-4) \quad x_0 = 5 \quad J = (4.5, 5.5) .$$

Remark: from problem (1) you know the interval J can be larger; Prof. G made J smaller than it can be to ~~mark~~ ^{make} part (i) easier for you.

You might find it easier to do problems (a) - (i) in a different order. Just do what you find easiest.

On parts (a) - (i), use ideas from only Sections 11.10 and 11.11, i.e., use only:

- the definition of Taylor polynomial
- the definition of Taylor series
- the theorem/error-estimate on the N^{th} -Remainder term for Taylor polynomials.

Do **NOT** use a known Taylor Series (i.e., do not use methods from Section 11.9, this was problem 1).

- a. Find the following. Note the first column are functions of x and the second column are numbers.

$f^{(0)}(x) = \ln(x-4)$	$f^{(0)}(x_0) = \ln 1 = 0$
$f^{(1)}(x) = (x-4)^{-1}$	$f^{(1)}(x_0) = 1$
$f^{(2)}(x) = -(x-4)^{-2}$	$f^{(2)}(x_0) = -1$
$f^{(3)}(x) = +2(x-4)^{-3}$	$f^{(3)}(x_0) = +2$

- b. Find N^{th} -order Taylor polynomial of $y = f(x)$ about x_0 in OPEN form for $N = 0, 1, 2$.

$P_0(x) = 0$
$P_1(x) = (x-5)^1$
$P_2(x) = (x-5)^1 - \frac{1}{2}(x-5)^2$

\rightarrow $n \geq 1$: see pattern?

$$f^{(n)}(x) = (-1)^{n-1} \cdot (n-1)! \cdot (x-4)^{-n}$$

$$C_n = \frac{f^{(n)}(5)}{n!} = (-1)^{n-1} \frac{(n-1)!}{n!} (5-4)^{-n} = \frac{(-1)^{n-1}}{n}$$

$$\frac{(n-1)!}{n!} = \frac{(n-1)!}{(n-1)! \cdot n} = \frac{1}{n}$$

- c. Find the Taylor series of $y = f(x)$ about x_0 in OPEN form.

$$P_{\infty}(x) = (x-5) - \frac{1}{2}(x-5)^2 + \frac{1}{3}(x-5)^3 - \frac{1}{4}(x-5)^4 +/\dots$$

- d. Find the Taylor series of $y = f(x)$ about x_0 in CLOSED form.

$$P_{\infty}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-5)^n$$

- e. Find the n^{th} Taylor coefficient of $y = f(x)$ about x_0 .

$$c_n = \frac{(-1)^{n-1}}{n} \quad \text{for } n \geq 1 \quad \text{and } c_0 = 0.$$

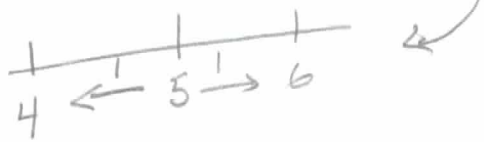
- f. Find the interval I of convergence of the Taylor series $y = f(x)$ about x_0 . Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

$$I = (+4, 6]$$

Ratio test

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} \cdot \frac{n}{1} = 1.$$

to abs. conv. when $|x-5| < 1$, ie $4 < x < 6$



Check endpoints

$$x=6 : \sum \frac{(-1)^{n-1}}{n} (6-5)^n = \sum \frac{(-1)^{n-1}}{n}$$

conditionally conv.

$$x=4 : \sum \frac{(-1)^{n-1}}{n} (4-5)^n = \sum \frac{(-1)^{n-1} (-1)^n}{n}$$

$$= \sum \frac{(-1)^{2n} (-1)^1}{n} = -\sum \frac{1}{n} = -\sum \frac{1}{n}$$

divg., harmonic series

$(4.5, 5.5)$
 $\parallel \leftarrow$ given.

- g. Consider the given interval J and fix an $N \in \mathbb{N}$. Find a good upper bound for the maximum of $|f^{(N+1)}(c)|$ on the interval J . Your answer can have an N in it but it cannot have an: x, x_0, c . (Note that J is a subset of I but Prof. G. might have picked a smaller J than I to make the problem easier.)

$$\max_{c \in J} |f^{(N+1)}(c)| \leq (2^{N+1})(N!)$$

$f^{(N+1)}(x) \stackrel{\text{part a}}{=} \frac{(-1)^N (N)!}{(x-4)^{N+1}}$

$4.5 < c < 5.5$
 $\frac{1}{2} = .5 < c-4 < 1.5 = \frac{3}{2}$

$\max_{c \in J} |f^{(N+1)}(c)| = \max_{4.5 < c < 5.5} \frac{N!}{|x-4|^{N+1}} = \frac{N!}{(\frac{1}{2})^{N+1}} = (2^{N+1})(N!)$

- h. Consider the given interval J and fix an $N \in \mathbb{N}$. For each $x \in J$, find a good upper bound for the maximum of $|R_N(x)|$. Your answer can have an N and x in it but it cannot have an: x_0, c .

$$|R_N(x)| \leq \frac{1}{N+1}$$

$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} (x-5)^{N+1} \right|$ and $\begin{cases} 4.5 < x < 5.5 \\ -\frac{1}{2} < x-5 < \frac{1}{2} \\ |x-5| < \frac{1}{2} \end{cases}$

$\stackrel{\text{part g}}{\leq} \frac{(2^{N+1}) N!}{(N+1)!} \left(\frac{1}{2}\right)^{N+1}$

$= \frac{N!}{(N!)(N+1)} \frac{2^{N+1}}{2^{N+1}} = \frac{1}{N+1}$

- i. Carefully show that $f(x) = P_\infty(x)$ for each x in the given interval J by using part (h) and showing that $\lim_{N \rightarrow \infty} |R_N(x)| = 0$ for each $x \in J$.

$$x \in J = (4.5, 5.5)$$

$$4.5 < x < 5.5$$

$$-\frac{1}{2} < x - 5 < \frac{1}{2}$$

$$|x - 5| < \frac{1}{2}$$

$$0 \leq \lim_{N \rightarrow \infty} |R_N(x)| \stackrel{\text{(part h)}}{\leq} \lim_{N \rightarrow \infty} \frac{1}{N+1} = 0$$

$$\text{So } \lim_{N \rightarrow \infty} |R_N(x)| = 0$$