

MARK BOX		
PROBLEM	POINTS	
1 a-k	35	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
%	100	

NAME: Key

PIN: \_\_\_\_\_

**INSTRUCTIONS:**

- (1) To receive credit you must:
  - (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that just appears;** such explanations help with partial credit
  - (b) if a line/box is provided, then:
    - show your work BELOW the line/box
    - put your answer on/in the line/box
  - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
- (3) You may **not** use a calculator, books, personal notes.
- (4) During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
- (5) If you do not make at least 17.5 out of the 35 points on Problem 1, then your score for the entire exam will be whatever you made on Problem 1.
- (6) This exam covers (from *Calculus* by Stewart, 6<sup>th</sup> ed., ET): 11.2–11.7 .

**Honor Code Statement**

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University's Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the above Instructions.

Signature : \_\_\_\_\_

Source: Spring 2010 Exam 2 #1 & 2.

1. Fill-in-the blanks/boxes. All series  $\sum$  are understood to be  $\sum_{n=1}^{\infty}$ .

1a.  $n^{\text{th}}$ -term test for an arbitrary series  $\sum a_n$ .

If  $\lim_{n \rightarrow \infty} a_n \neq 0$  or  $\lim_{n \rightarrow \infty} a_n$  does not exist, then  $\sum a_n$  diverges.

1b. Geometric Series where  $-\infty < r < \infty$ . The series  $\sum r^n$

- converges if and only if  $|r|$   $< 1$
- diverges if and only if  $|r|$   $\geq 1$

1c.  $p$ -series where  $0 < p < \infty$ . The series  $\sum \frac{1}{n^p}$

- converges if and only if  $p$   $> 1$
- diverges if and only if  $p$   $\leq 1$

1d. Integral Test for a positive-termed series  $\sum a_n$  where  $a_n \geq 0$ .

Let  $f: [1, \infty) \rightarrow \mathbb{R}$  be so that

- $a_n = f(\underline{n})$  for each  $n \in \mathbb{N}$
- $f$  is a positive function
- $f$  is a decreasing function
- $f$  is a continuous function.

*nonincreasing is also ok*

Then  $\sum a_n$  converges if and only if  $\int_1^{\infty} f(x) dx$  converges.

1e. Comparison Test for a positive-termed series  $\sum a_n$  where  $a_n \geq 0$ . (Fill in the blanks with  $a_n$  and/or  $b_n$ .)

- If  $0 \leq a_n \leq b_n$  for all  $n \in \mathbb{N}$  and  $\sum b_n$  converge, then  $\sum a_n$  converge.
- If  $0 \leq b_n \leq a_n$  for all  $n \in \mathbb{N}$  and  $\sum b_n$  diverge, then  $\sum a_n$  diverge.

1f. Limit Comparison Test for a positive-termed series  $\sum a_n$  where  $a_n \geq 0$ .

Let  $b_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ .

If 0  $< L < \infty$ , then  $\sum a_n$  converges if and only if  $\sum b_n$  converges

1g. Ratio and Root Tests for arbitrary-termed series  $\sum a_n$  with  $-\infty < a_n < \infty$ .

Let  $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  or  $\rho = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$ .

- If  $\rho$   $< 1$  then  $\sum a_n$  converges absolutely.
- If  $\rho$   $> 1$  then  $\sum a_n$  diverges.
- If  $\rho$   $= 1$  then the test is inconclusive.

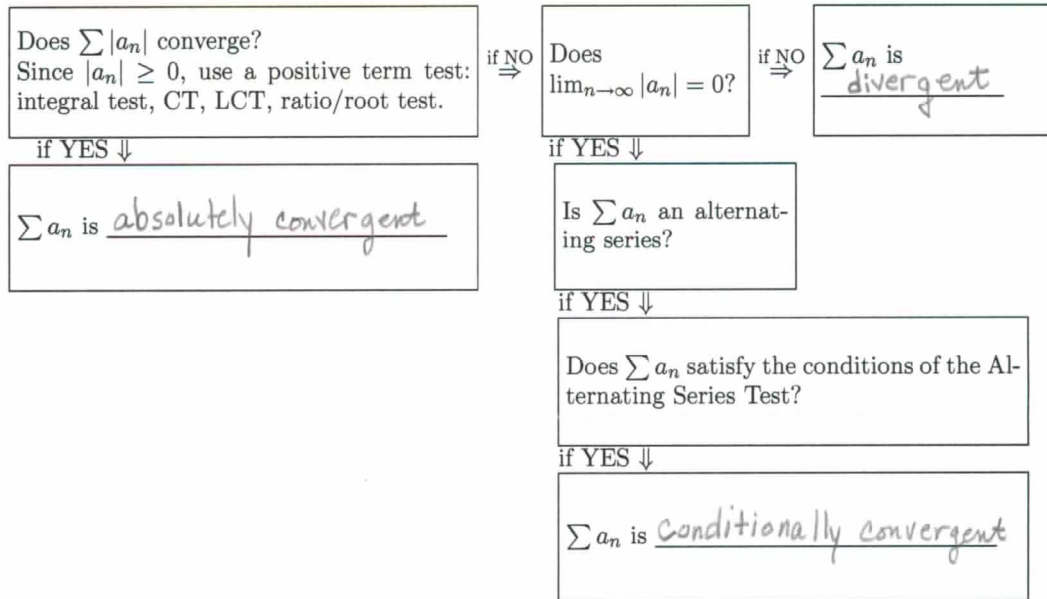
1h. Alternating Series Test for an alternating series  $\sum (-1)^n a_n$  where  $a_n > 0$  for each  $n \in \mathbb{N}$ .

If

- $a_n$   $>$   $a_{n+1}$  for each  $n \in \mathbb{N}$
- $\lim_{n \rightarrow \infty} a_n =$  0

then  $\sum (-1)^n a_n$  converges

- 1i. By definition, for an arbitrary series  $\sum a_n$ , (fill in the blanks with converges or diverges).
- $\sum a_n$  is absolutely convergent if and only if  $\sum |a_n|$  converges
  - $\sum a_n$  is conditionally convergent if and only if  $\sum a_n$  converges and  $\sum |a_n|$  diverges
  - $\sum a_n$  is divergent if and only if  $\sum a_n$  diverges
- 1j. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series  $\sum_{n=17}^{\infty} a_n$  is: absolutely convergent, conditional convergent, or divergent.



- 1k. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

- T      F      If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .
- T       F      If  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\sum a_n$  converges
- T      F      If  $\sum a_n$  converges and  $\sum b_n$  converge, then  $\sum (a_n + b_n)$  converges.
- T       F      If  $\sum (a_n + b_n)$  converges, then  $\sum a_n$  converges and  $\sum b_n$  converge.
- T       F      If  $S_N = \sum_{n=1}^N r^n$ , then  $S_N = \frac{r - r^{N+1}}{1 - r}$ , for when  $r \neq 1$  since we don't like to divide by zero

$$S_N = r^1 + r^2 + r^3 + \dots + r^N$$

$$r S_N = \phantom{r^1} + r^2 + r^3 + \dots + r^N + r^{N+1}$$


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$$(1-r) S_N = r^1 - r^{N+1} \Rightarrow S_N = \frac{r - r^{N+1}}{1 - r}$$

Source a warm up "confidence building" problem.

2. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$



absolutely convergent



conditionally convergent



divergent

Check abs. conv.

$$\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2}$$

← p-series  
 $p = 2 > 1$   
conv.

Since  $\sum \left| \frac{(-1)^n}{n^2} \right|$  conv.,  $\sum \frac{(-1)^n}{n^2}$  is abs. conv.

Source similar to an example in class.

3. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{4n^7 - n^6 + 21}{11n^8 - n + 17}$$

absolutely convergent

~~conditionally convergent~~ b/c it's a positive-termed series

divergent

You may use, without showing, that for  $n \geq 1$

$$a_n := \frac{4n^7 - n^6 + 21}{11n^8 - n + 17} > 0.$$

Hint. What would be easier: CT or LCT?

Thinking Land

$$\frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \approx \frac{4n^7}{11n^8} = \frac{4}{11} \cdot \frac{1}{n}$$

So let's do LCT w/  $b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \cdot \frac{n}{1} = \frac{4n^8 - n^7 + 21n}{11n^8 - n + 17} \xrightarrow{n \rightarrow \infty} \frac{4}{11} < L.$$

So  $\sum a_n$  &  $\sum b_n$  "do the same thing"

$\sum b_n = \sum \frac{1}{n}$  divg (p-series,  $p=1 \leq 1$  ... or can say "harmonic series")

So  $\sum a_n$  divg.

Comments on # 3.

The calculations on the following page shows that if  $n \in \mathbb{N} = \{1, 2, 3, \dots\}$  then

$$\frac{4}{11n} \leq \frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \iff n = 1$$

but

$$\frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \leq \frac{4}{11n} \iff n \geq 2.$$

Note  
 $n \geq 1$

$$\frac{4}{11n} \stackrel{(*)}{<} \frac{4n^7 - n^6 + 21}{11n^8 - n + 17} \rightarrow \text{both positive for } n \geq 1$$

$$\Leftrightarrow 4(11n^8 - n + 17) < 11n(4n^7 - n^6 + 21)$$

$$\Leftrightarrow 44n^8 - 4n + 68 < 44n^8 - 11n^7 + 231n$$

$$\Leftrightarrow 0 < -11n^7 + 235n - 68.$$

Let  $f(x) = -11x^7 + 235x - 68$

$$f'(x) = -77x^6 + 235$$

$$f'(x_0) = 0 \Leftrightarrow x_0 = \sqrt[6]{\frac{235}{77}} \approx 1.2$$

$$\begin{array}{c} f' > 0 \quad | \quad f' < 0 \\ \hline x_0 \end{array}$$

$$f(x_0) \approx 175.$$

$$\lim_{x \rightarrow 0} f(x) = -\infty$$

$$f(1) = 156$$

$$f(2) = -1006$$

$$\bullet \overline{18} = \frac{4}{11(2)} \quad \text{but} \quad \frac{4(2)^7 - 2^6 + 21}{11(2)^8 - 2 + 17} = \frac{469}{2831} \approx 0.166$$

Source Exam 2 # 5 Spring 2010

4. Let

$$a_n = \frac{n!}{(2n-1)!}$$

4a. Find an expression for  $\frac{a_{n+1}}{a_n}$  that does NOT have a factorial sign (that is a ! sign) in it.

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n(2n+1)} \quad \text{or} \quad \frac{n+1}{4n^2+2n}$$

4b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

- absolutely convergent  
 conditionally convergent  
 divergent

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n-1)!}$$

$$a_{n+1} = \frac{(n+1)!}{(2(n+1)-1)!} = \frac{(n+1)!}{(2n+2-1)!} = \frac{(n+1)!}{(2n+1)!} \quad \rightarrow \text{collect up like terms}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2n+1)!} \cdot \frac{(2n-1)!}{n!} = \frac{(n+1)!}{n!} \cdot \frac{(2n-1)!}{(2n+1)!} = \frac{n!(n+1)}{n!} \cdot \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)}$$

$$= \frac{(n+1)}{(2n)(2n+1)} = \frac{n+1}{4n^2+2n} \quad \begin{matrix} \uparrow \\ \text{divide top \& bottom by } n^2 \end{matrix} = \frac{\frac{1}{n} + \frac{1}{n^2}}{4 + \frac{2}{n}} \quad \xrightarrow{n \rightarrow \infty} \frac{0+0}{4+0} = 0 = \rho < 1$$

By Ratio Test, since  $\rho < 1$ ,  $\sum (-1)^n a_n$  is abs. conv.



Source: Serious Series Problem # 8.

5. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{3n^2 + 8n - 1}$$

absolutely convergent

conditionally convergent

divergent

Check abs. conv.

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{3n^2 + 8n - 1} = \frac{1}{3} \neq 0.$$

So by  $n^{\text{th}}$  term test for divg.,  $\sum \frac{n^2 + 1}{3n^2 + 8n - 1}$  divg.

Check cond. conv.

$$\lim_{n \rightarrow \infty} (-1)^n \frac{n^2 + 1}{3n^2 + 8n - 1} \text{ DNE (osc.)}$$

So by  $n^{\text{th}}$  term test for divg.,  $\sum (-1)^n \frac{n^2 + 1}{3n^2 + 8n - 1}$  divg

# Homework problem § 11.7 # 19

6. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$$

absolutely convergent  
 conditionally convergent  
 divergent

19. Let  $f(x) = \frac{\ln x}{\sqrt{x}}$ . Then  $f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$  when  $\ln x > 2$  or  $x > e^2$ , so  $\frac{\ln n}{\sqrt{n}}$  is decreasing for  $n > e^2$ .

By l'Hospital's Rule,  $\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1/n}{1/(2\sqrt{n})} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0$ , so the series  $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$  converges by the

Alternating Series Test.

Now need to consider

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}} \quad \underline{\underline{\text{and}}} \quad \text{Let } a_n = \frac{\ln n}{n^{1/2}}$$

For  $n \geq 3$

$$\frac{1}{n^{1/2}} \leq \frac{\ln n}{n^{1/2}}$$

$$\sum_{n=3}^{\infty} \frac{1}{n^{1/2}} = \infty \quad (\text{p-series, } p = \frac{1}{2} < 1) \quad \text{So}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}} = \infty \quad \text{by cT.}$$

Limit Comparison Test  
(LCT)

Let  $b_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ .

If  $0 < L < \infty$ , then  $[\sum a_n \text{ conv.} \iff \sum b_n \text{ conv.}]$

(you DO need to memorize this one)

If  $L = 0$ , then  $[\sum b_n \text{ conv.} \implies \sum a_n \text{ conv.}]$

(you do not have to memorize this one)

**\***  $\implies$  If  $L = \infty$ , then  $[\sum b_n \text{ divg.} \implies \sum a_n \text{ divg.}]$

(you do not have to memorize this one)

Let  $b_n = \frac{1}{\sqrt{n}}$ . Note  $\sum b_n$  divg b/c p-series,  $p = 1/2 < 1$ .

$$\frac{a_n}{b_n} = \frac{\ln n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \ln n \xrightarrow{n \rightarrow \infty} \infty$$

So  $\sum a_n = \sum \frac{\ln n}{\sqrt{n}}$  divg.

NAY #1

NAY #2

Similar to Exam 2 # 9 Spring 2010.

7. Geometric Series. (On this page, you should do basic algebra but you do NOT have to do any fancy arithmetic (eg, just making up numbers, you can leave  $(\frac{17}{18})^{171}$  as just that.) Let, for  $N \geq 51$ ,

$$s_N = \sum_{n=51}^N 2 \frac{3^{n+1}}{5^n}.$$

- 7a. Do some algebra to write  $s_N$  as  $\sum_{n=51}^N c r^n$  for an appropriate constant  $c$  and ratio  $r$ .

$$s_N = \sum_{n=51}^N 6 \left(\frac{3}{5}\right)^n$$

$$2 \cdot \frac{3^{n+1}}{5^n} = \frac{2 \cdot 3^n \cdot 3}{5^n} = 2 \cdot 3 \cdot \frac{3^n}{5^n} = 6 \left(\frac{3}{5}\right)^n$$

- 7b. Using the method from class (rather than some formula), find an expression for  $s_N$  in closed form (i.e. without a summation  $\sum$  sign nor some dots ...).

$$s_N = 15 \left[ \left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1} \right]$$

$$s_N = 6 \left[ \left(\frac{3}{5}\right)^{51} + \left(\frac{3}{5}\right)^{52} + \left(\frac{3}{5}\right)^{53} + \dots + \left(\frac{3}{5}\right)^N \right]$$

$$\frac{3}{5} s_N = 6 \left[ \left(\frac{3}{5}\right)^{52} + \left(\frac{3}{5}\right)^{53} + \dots + \left(\frac{3}{5}\right)^N + \left(\frac{3}{5}\right)^{N+1} \right]$$

subtract

$$\frac{2}{5} s_N \stackrel{\text{not}}{=} \left(1 - \frac{3}{5}\right) s_N = 6 \left[ \left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1} \right]$$

$$s_N = \frac{5}{2} \cdot 6 \left[ \left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1} \right]$$

- 7c. Does  $\sum_{n=51}^{\infty} 2 \frac{3^{n+1}}{5^n}$  converge or diverge? If it converges, find its sum. Justify your answer.

$$\sum_{n=51}^{\infty} 2 \frac{3^{n+1}}{5^n} = 15 \left(\frac{3}{5}\right)^{51} \quad \text{converges}$$

$$\lim_{N \rightarrow \infty} s_N = \lim_{N \rightarrow \infty} 15 \left[ \left(\frac{3}{5}\right)^{51} - \left(\frac{3}{5}\right)^{N+1} \right]$$

$$= 15 \left[ \left(\frac{3}{5}\right)^{51} - 0 \right]$$

since  $|\frac{3}{5}| < 1$