MARK BOX		
PROBLEM	POINTS	
1(23)/2(10)/3(7)	40	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
take home	10	
	150	

NAME (legibly printed): Sol'n

class PIN: 17

If you do not make at least a 50% (i.e. at least 20 of the 40 possible points) on problems 1, 2, and 3, then you will recieve a zero on the remainder of the exam. You were warned.

INSTRUCTIONS:

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) if a line/box is provided, then:
    — show you work BELOW the line/box
    — put your answer on/in the line/box
  (c) if no such line/box is provided, then box your answer

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) You may **not** use a calculator, books, personal notes.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from *Calculus* by Stewart, 6^{th} ed., ET): 7.1 - 7.5, 7.8, 11.1 - 11.8, 6.1 - 6.3, 10.3, 10.4. .

Problem Inspiration: See the answer key.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.
Furthermore, I have not only read but will also follow the above Instructions.

Signature : Prof. Maria Girardi

Problem Source: Exam 1 #1 Spring 2010

1. Fill in the blanks.

- $\int \frac{du}{u} = $ ln $|u| + C$
- $\int \cos u\, du = $ sin u $+C$
- $\int \sin u\, du = $ $-\cos u$ $+C$
- $\int \tan u\, du = $ $\ln|\sec u|$ or $-\ln|\cos u|$ $+C$
- $\int \cot u\, du = $ $-\ln|\csc u|$ or $\ln|\sin u|$ $+C$
- $\int \sec u\, du = $ $\ln|\sec u + \tan u|$ or $-\ln|\sec u - \tan u|$ $+C$
- $\int \csc u\, du = $ $-\ln|\csc u + \cot u|$ or $\ln|\csc u - \cot u|$ $+C$
- $\int \sec^2 u\, du = $ $\tan u$ $+C$
- $\int \sec u \tan u\, du = $ $\sec u$ $+C$
- $\int \csc^2 u\, du = $ $-\cot u$ $+C$
- $\int \csc u \cot u\, du = $ $-\csc u$ $+C$
- If a is a contant and $a > 0$ then $\int \frac{1}{a^2+u^2}\, du = $ $\frac{1}{a} \tan^{-1} \frac{u}{a}$ $+C$
- If a is a contant and $a > 0$ then $\int \frac{1}{\sqrt{a^2-u^2}}\, du = $ $\sin^{-1} \frac{u}{a}$ $+C$
- If a is a contant and $a > 0$ then $\int \frac{1}{u\sqrt{u^2-a^2}}\, du = $ $\frac{1}{a} \sec^{-1} \frac{|u|}{a}$ $+C$
- Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do long division
- Integration by parts formula: $\int u\, dv = $ $uv - \int v\, du$
- Trig substitution: (recall that the *integrand* is the function you are integrating) if the integrand involves $a^2 + u^2$, then one makes the substitution $u = $ $a \tan\theta$
- Trig substitution: if the integrand involves $a^2 - u^2$, then one makes the substitution $u = $ $a \sin\theta$
- Trig substitution: if the integrand involves $u^2 - a^2$, then one makes the substitution $u = $ $a \sec\theta$
- trig formula ... your answer should involve trig functions of θ, and not of 2θ: $\sin(2\theta) = $ $2\sin\theta\cos\theta$.
- trig formula ... your answer should have $\cos(2\theta)$ in it: $\cos^2(\theta) = \frac{1}{2}($ $1 + \cos 2\theta$ $)$.
- trig formula ... your answer should have $\cos(2\theta)$ in it: $\sin^2(\theta) = \frac{1}{2}($ $1 - \cos 2\theta$ $)$.
- trig formula ... since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween tangent (i.e., tan) and secant (i.e., sec) is $1 + \tan^2\theta = \sec^2\theta$.

Problem Source: Spring 2010 Exam 2 # 1 and 2

2. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

- n^{th}**-term test** for an arbitrary series $\sum a_n$.
  If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges.

- **Geometric Series** where $-\infty < r < \infty$. The series $\sum r^n$
  - converges if and only if $|r|$ < 1
  - diverges if and only if $|r|$ ≥ 1

- p**-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$
  - converges if and only if p > 1
  - diverges if and only if p ≤ 1

- **Integral Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
  Let $f\colon [1,\infty) \to \mathbb{R}$ be so that
  - $a_n = f($ n $)$ for each $n \in \mathbb{N}$
  - f is a positive function
  - f is a continuous function
  - f is a decreasing (or nonincreasing) function.

  Then $\sum a_n$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.

- **Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
  - If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
  - If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

- **Limit Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
  Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.
  If 0 $< L <$ ∞, then $\sum a_n$ converges if and only if $\sum b_n$ converges

- **Ratio and Root Tests** for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
  Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.
  - If ρ < 1 then $\sum a_n$ converges.
  - If ρ > 1 then $\sum a_n$ diverges.
  - If ρ $= 1$ then the test is inconclusive.

- **Alternating Series Test** for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.
  If
  - a_n $>$ a_{n+1} for each $n \in \mathbb{N}$
  - $\lim_{n\to\infty} a_n =$ 0

  then $\sum(-1)^n a_n$ converges

- By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).
  - $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
  - $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
  - $\sum a_n$ is divergent if and only if $\sum a_n$ diverges
- Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=17}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

Does $\sum |a_n|$ converge?
Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$

Does $\lim_{n\to\infty} |a_n| = 0$?

if NO $\Rightarrow$

$\sum a_n$ is divergent

if YES $\Downarrow$ (from "Does $\sum |a_n|$ converge?")

$\sum a_n$ is absolutely convergent

if YES $\Downarrow$ (from "Does $\lim_{n\to\infty} |a_n| = 0$?")

Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is conditionally convergent

Problem Source: Exam 3, #1& 2, Spring 2010.

3. Fill-in-the-blanks/boxes, **using words and/or formula involving *some of*:**

perpendicular, parallel, 2 , π , radius , radius$_{\text{big}}$, radius$_{\text{little}}$, average radius , height , thickness.

►. Disk/Washer Method

Let's say you revolve some region in the xy-plane around an axis of revolution so you get a solid of revolution. Next you want to find the volume of this solid of revolution using the disk or washer method.

- You should partition the coordinate axis (i.e., the x-axis or the y-axis) that is parallel to the axis of revolution.

- If you use the **disk method**, then the volume of a typical disk is:

$\pi\,(\text{radius})^2\,(\text{height})$.

- If you use the **washer method**, then the volume of a typical washer is:

$\pi\,(\text{rad}_{\text{big}})^2(\text{height}) - \pi\,(\text{rad}_{\text{little}})^2(\text{height}) \overset{\text{or}}{=} \pi\left[(\text{rad}_{\text{big}})^2 - (\text{rad}_{\text{little}})^2\right](\text{height})$.

- If you partition the z-axis, the Δz = height.

►. Shell Method

Let's say you revolve some region in the xy-plane around an axis of revolution so you get a solid of revolution. Next you want to find the volume of this solid of revolution using the shell method.

- You should partition the coordinate axis (i.e., the x-axis or the y-axis) that is perpendicular to the axis of revolution.

- If you use the **shell method**, then the volume of a typical shell is:

$2\pi\,(\text{average radius})(\text{height})(\text{thickness})$.

- If you partition the z-axis, the Δz = $\text{thickness} \overset{\text{or}}{=} \text{radius}_{\text{big}} - \text{radius}_{\text{little}}$.

Problem inspiration: Spring 2010 Ex 1 # 2.

4. $\boxed{\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C}$

Hint: a formula from problem 1 is handy.

$$\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{x}{2} + \frac{1}{2}\cdot\frac{1}{2}\int \cos 2x \,(2\,dx)$$

$$= \frac{x}{2} + \frac{1}{4}\sin(2x) + C$$

Problem Inspiration : Spring 2010, Exam 1, #3

5. $\int x^{17}\ln(x)\,dx = \dfrac{x^{18}\ln x}{18} - \dfrac{x^{18}}{(18)^2} + C$

Hint: $D_x \ln x = \frac{1}{x}$ and $\int \frac{dx}{x} = \ln|x| + C$.

So $y = \ln x$ is hard to integrate but easy to differeniate so which method should you try? Parts

with u = that hard function to integrate.

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^{17}\,dx \\ du = \frac{1}{x}\,dx & V = \frac{x^{18}}{18} \end{array}}$$

$$\int x^{17}\ln(x)\,dx = \frac{x^{18}\ln x}{18} - \frac{1}{18}\int \frac{x^{18}}{x}\,dx$$

$$= \frac{x^{18}\ln x}{18} - \frac{1}{18}\int x^{17}\,dx$$

$$= \frac{x^{18}\ln x}{18} - \frac{1}{18}\,\frac{x^{18}}{18} + C$$

ps. $18^2 = 324$.

Problem Inspiration: Spring 2010, Ex 1, # 6 (but harder)

6. $\boxed{\int \sec^5 x \tan^5 x \, dx = \frac{\sec^9 x}{9} - \frac{2\sec^7 x}{7} + \frac{\sec^5 x}{5} + C}$

Hint: formulas from problem 1 are handy.

$\boxed{\begin{aligned} s &= \sec x \\ ds &= \sec x \tan x \, dx \end{aligned}}$ $\quad$ $\boxed{\cancel{\begin{aligned} t &= \tan x \\ dt &= \sec^2 x \, dx \end{aligned}}}$

$$\int \sec^5 x \tan^5 x \, dx = \int \sec^4 x \tan^4 x \; \boxed{\sec x \tan x \, dx}$$

$$= \int \sec^4 x \, (\tan^2 x)^2 \; \boxed{\sec x \tan x \, dx}$$

$$= \int \sec^4 x \, (\sec^2 x - 1)^2 \; \boxed{\sec x \tan x \, dx}$$

$$= \int s^4 (s^2 - 1)^2 \, ds$$

$$= \int s^4 (s^4 - 2s^2 + 1) \, ds$$

$$= \int (s^8 - 2s^6 + s^4) \, ds$$

$$= \frac{s^9}{9} - \frac{2s^7}{7} + \frac{s^5}{5} + C$$

Problem Source: Textbook, § 7.4, # 15

7. $$\boxed{\int \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = x + \ln|x| - \frac{2}{x} - \ln|x-2| \quad + C}$$

PFD

Hint: Do we have (Strictly) Bigger Bottoms ?

NO so need to do long division .... but it's easy to "fake" long division on this one (lucky us)

$$\frac{x^3-2x^2-4}{x^3-2x^2} = \frac{x^3-2x^2}{x^3-2x^2} + \frac{-4}{x^3-2x^2} = 1 + \frac{-4}{x^3-2x^2}$$

Find PFD for $\frac{-4}{x^3-2x^2} = \frac{-4}{x^2(x-2)} = \frac{-4}{\underbrace{(x-0)^2}_{(\text{linear term})^2}\underbrace{(x-2)}_{(\text{linear term})^1}}$

$$\frac{-4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}$$

$\Rightarrow \boxed{-4 = Ax(x-2)+B(x-2)+Cx^2}$ $\xrightarrow{x=0}$ $-4 = -2B \Rightarrow \boxed{B=2}$

$\xrightarrow{x=2}$ $-4 = C\cdot 2^2 \Rightarrow \boxed{C=-1}$

equate coeff.

$x^2: 0 = A + C \xrightarrow{C=-1} \boxed{A=1}$

$x^1: 0 = -2A + B$

$x^0: -4 = -2B$

$$\int \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int \left[1 + \frac{1}{x} + \frac{2}{x^2} + \frac{-1}{x-2}\right] dx$$

$$\Downarrow$$

$$\int 2x^{-2}\,dx = \frac{2x^{-1}}{-1} + C$$

Problem Inspiration: Textbook, §7.3, #24

8a. Complete the square:

$$x^2 - 6x + 13 = \boxed{(x-3)^2 + 4}$$

$$x^2 - 6x + 13 = \underbrace{(x^2 - 6x + 9)}_{① \uparrow \text{ think}} \overset{②}{} \underbrace{- 9 + 13}_{③}$$

$$= (x-3)^2 + 4$$

8b.

$$\int \frac{1}{\sqrt{x^2 - 6x + 13}}\, dx = \boxed{\ln\left| \frac{\sqrt{x^2 - 6x + 13}}{2} + \frac{x-3}{2} \right| + C}$$

Have $(x-3)^2 + 2^2$

$\quad\;\; \| \qquad\quad \|$

$\quad\;\; u^2 \;\; + \; a^2 \quad \Rightarrow \quad u = a \tan\theta$

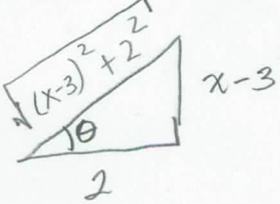
$$x - 3 = 2\tan\theta \Rightarrow x = 3 + 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$$

$$x^2 - 6x + 13 = (x-3)^2 + 2^2 = (2\tan\theta)^2 + 2^2 = 4(\tan^2\theta + 1) = 4\sec^2\theta$$

$$\int \frac{dx}{\sqrt{x^2 - 6x + 13}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\tan\theta = \frac{x-3}{2} \Rightarrow \sec\theta = \frac{\sqrt{(x-3)^2 + 2^2}}{2} = \frac{\sqrt{x^2 - 6x + 13}}{2}$$



Problem Inspiration : 09 Fall final #5

9. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}}$$

- [ ] ~~absolutely convergent~~ LCT w/ $b_n = \frac{1}{n}$
- [x] conditionally convergent then use AST
- [ ] divergent

Warning: there is a square root in the denominator ... many of you overlooked $\sqrt{\ }$'s on Exams ... see it?.

Abs. Conv? Consider $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right| = \sum \frac{1}{\sqrt{(n+2)(n+7)}}$

Thinking Land $a_n = \frac{1}{\sqrt{(n+2)(n+7)}} \overset{n \text{ big}}{\approx} \frac{1}{\sqrt{n \cdot n}} = \frac{1}{n} = b_n$

LCT. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{(n+2)(n+7)}} = 1$

more details $\leadsto$ $= \lim_{n\to\infty} \sqrt{\left[\frac{n^2}{(n+2)(n+7)}\right]} = \sqrt{1} = 1 \equiv L$, $0 < L < \infty$

so $\sum b_n$ & $\sum a_n$ do the same thing. $\sum b_n$ divg (harmonic series)

so $\sum \left| (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \right|$ dvg so <u>not</u> abs. conv.

<u>Cond. Conv?</u> Let's use AST w/ $0 \le u_n = \frac{1}{\sqrt{(n+2)(n+7)}}$

(1) u_n dec., ie $u_n > u_{n+1}$? yes clear.

(2) $\lim_{n\to\infty} \frac{1}{\sqrt{(n+2)(n+7)}} = 0$ ☺

so, by AST, $\sum (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}}$ conv.

Problem Inspiration : Spring 2010, Ex 2, # 5

10. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=8}^{\infty} (-1)^n \frac{(n+1)!}{(2n)!}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

But before you get started .... let

$$a_n = \frac{(n+1)!}{(2n)!}.$$

Then $a_{n+1} = \dfrac{((n+1)+1)!}{(2(n+1))!} = \dfrac{(n+2)!}{(2n+2)!}$

Next, simplify $\frac{a_{n+1}}{a_n}$ so that it has NO factorial sign (that is a ! sign) in it.

$\dfrac{a_{n+1}}{a_n} = \dfrac{n+2}{(2n+1)(2n+2)}$ or $\dfrac{n+2}{4n^2+6n+2}$

Ok, now you should be ready to finish off the problem and check the correct box above.

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)!}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)!} = \frac{(n+2)!}{(n+1)!} \cdot \frac{(2n)!}{(2n+2)!} = \frac{(n+1)!\,(n+2)}{(n+1)!} \cdot \frac{(2n)!}{(2n)!\,(2n+1)(2n+2)}$$

Abs. Conv? Consider $\sum \frac{(n+1)!}{(2n)!}$ & use Ratio Test

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \overset{\text{above}}{=} \lim_{n\to\infty} \frac{n+2}{(2n+1)(2n+2)} = 0 < 1$$

$\Downarrow$ Ratio Test

Conv.

Problem Inspiration : Fall 09 Final # 7.

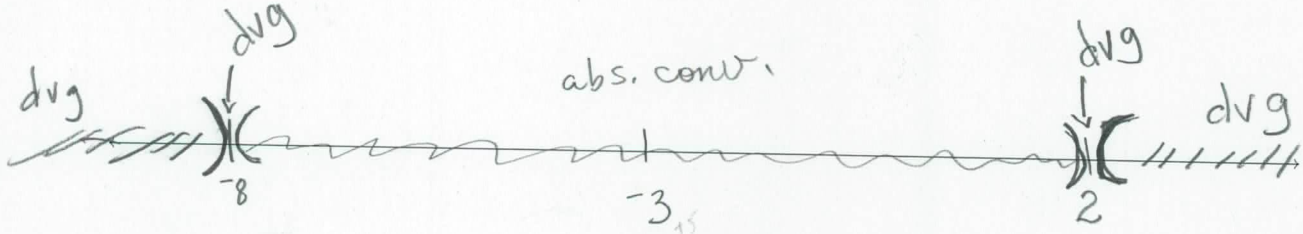
11. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{10^n}$$

The center is $x_0 =$ ___ -3 ___ and the radius of convergence is $R =$ ___ 5 ___.
As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any. Remember your absolute value signs! Algebra hint:

$$\left|\frac{(2x+6)^n}{10^n}\right| = \left|\left(\frac{2x+6}{10}\right)^n\right| = \left|\left(\frac{x+3}{5}\right)^n\right| = \left(\frac{|x+3|}{5}\right)^n.$$

Now, do you want to use the ratio or root test?



Root Test $\quad \rho = \lim_{n\to\infty} \left|\frac{(2x+6)^n}{10^n}\right|^{1/n} \underset{\text{above}}{=} \lim_{n\to\infty}\left(\left(\frac{|x+3|}{5}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{|x+3|}{5}$

$$= \frac{|x+3|}{5} \overset{?}{<} 1 \iff \boxed{|x - {}^-3| < 5}$$

endpts are $\quad -3 \pm 5 <^{2}_{-8}$

Check endpts

$x=2 \quad \sum \frac{(2x+6)^n}{10^n} \underset{x=2}{=} \sum \frac{10^n}{10^n} = \sum 1 = \infty$, big time.

$x=-8 \quad \sum \frac{(2x+6)^n}{10^n} \underset{x=-8}{=} \sum \frac{(-10)^n}{10^n} = \sum \frac{(-1)^n (10)^n}{(10)^n} = \sum (-1)^n$ dvg/osc

Fall 09, Final, #16 & 17
Fall 08, Exam 3, #7

$2x = x^2$
$0 = x^2 - 2x = x(x-2)$
$x = 0 \rightarrow (0,0)$
$x = 2 \rightarrow (2,4)$

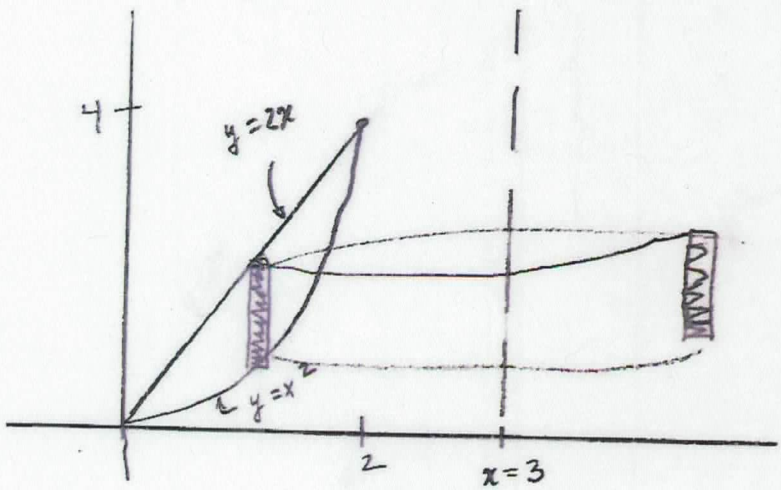
- Let R be the region in the first quadrant enclosed by $y = 2x$ and $y = x^2$.

12 ~~16~~. Express the area of R as integral(s) with respect to x.

Area = $\displaystyle\int_{x=0}^{x=2} \left[(2x) - (x^2) \right] dx$

13 ~~17~~. Using the shell method, express as integral(s) the volume of the solid generated by revolving R about the line $x = 3$.

Volume = $\displaystyle\int_{x=0}^{x=2} 2\pi (3-x)(2x-x^2)\, dx$



$V_{\text{typical shell}} = 2\pi \,(\text{avg. rad.})\,(\text{height})\,(\text{thickness})$

$\downarrow$

Δx