MARK BOX		
PROBLEM	POINTS	
a - j	30	
TOTAL	30	

NAME: Sol'n

class PIN:

INSTRUCTIONS:

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
(3) This exam covers (from *Calculus* by Stewart 6^{th} ed,ET): § 11.9, 11.10, 11.11 .

Problem Inspiration: just like the homework.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.
Furthermore, I have not only read but will also follow the above Instructions.
I hereby verify that I did NOT receive help from other people on this take-home exam problem.

Signature :

Due Friday April 16 by 1pm.
Either hand me your paper in class or
slip your paper under my office (LC 309C) door.

Just FYI

Homework on Taylor/Maclaurin Polynomials and Series

Part 1 — Fill in the box

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

A. In open form (i.e., with … and without a $\sum$-sign)

$$P_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N$$

B. In closed form (i.e., with a $\sum$-sign and without …)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

C. In open form (i.e., with … and without a $\sum$-sign)

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \dots$$

D. In closed form (i.e., with a $\sum$-sign and without …)

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

E. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{(N+1)}}$$ for some c between $\boxed{x}$ and $\boxed{x_0}$.

F. The formula for c_n is

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

FYI

Taylor/Maclaurin Polynomials and Series

Do parts (a) - (j) for the following problem.

$$f(x) = (4-x)^{-2} \qquad x_0 = 1 \qquad J = (0,2)\ .$$

You might find it easier to do problems (a) - (j) in a different order. Just do what you find easiest.

- On parts (a) - (i), use ideas from only Sections 11.10 and 11.11, i.e., use only:
  - the definition of Taylor polynominal
  - the definition of Taylor series
  - the theorem/error-estimate on the N^{th}-Remainder term for Talyor polyominals.

  Do **NOT** use a known Taylor Series (i.e., do not use methods from Section 11.9).
- On part (j), the very last part, use a known Taylor Series (as from the handout `Commonly Used Taylor Series`) and methods from Section 11.9.

a. Find the following. Note the first column are functions of x and the second column are numbers.

$f^{(0)}(x) = (4-x)^{-2} = 1!\,(4-x)^{-2}$	$f^{(0)}(x_0) = 1!\,(3)^{-2}$
$f^{(1)}(x) = 2\,(4-x)^{-3} = 2!\,(4-x)^{-3}$	$f^{(1)}(x_0) = 2!\,(3)^{-3}$
$f^{(2)}(x) = 2\cdot 3\,(4-x)^{-4} = 3!\,(4-x)^{-4}$	$f^{(2)}(x_0) = 3!\,(3)^{-4}$
$f^{(3)}(x) = 2\cdot 3\cdot 4\,(4-x)^{-5} = 4!\,(4-x)^{-5}$	$f^{(3)}(x_0) = 4!\,(3)^{-5}$
$f^{(4)}(x) = 2\cdot 3\cdot 4\cdot 5\,(4-x)^{-6} = 5!\,(4-x)^{-6}$	$f^{(4)}(x_0) = 5!\,(3)^{-6}$

b. Find N^{th}-order Taylor polynomial of $y = f(x)$ about x_0 in OPEN form for $N = 0, 1, 2, 3, 4$.

$P_0(x) = \frac{1}{3^2}$	$C_0 = \frac{1}{3^2}$	$= \frac{1}{3^2}$
$P_1(x) = \frac{1}{3^2} + \frac{2}{3^3}(x-1)$	$C_1 = \frac{2!}{3^3}$	$= \frac{2}{3^3}$
$P_2(x) = \frac{1}{3^2} + \frac{2}{3^3}(x-1) + \frac{3}{3^4}(x-1)^2$	$C_2 = \frac{1}{2!}\frac{3!}{3^4}$	$= \frac{3}{3^4}$
$P_3(x) = \frac{1}{3^2} + \frac{2}{3^3}(x-1) + \frac{3}{3^4}(x-1)^2 + \frac{4}{3^5}(x-1)^3$	$C_3 = \frac{1}{3!}\frac{4!}{3^5}$	$= \frac{4}{3^5}$
$P_4(x) = \frac{1}{3^2} + \frac{2}{3^3}(x-1) + \frac{3}{3^4}(x-1)^2 + \frac{4}{3^5}(x-1)^3 + \frac{5}{3^6}(x-1)^4$	$C_4 = \frac{1}{4!}\frac{5!}{3^6}$	$= \frac{5}{3^6}$

helpful but not necessary

c. Find the Taylor series of $y = f(x)$ about x_0 in OPEN form.

$$P_\infty(x) = \frac{1}{3^2} + \frac{2}{3^3}(x-1) + \frac{3}{3^4}(x-1)^2 + \frac{4}{3^5}(x-1)^3 + \frac{5}{3^6}(x-1)^4 + \cdots$$

d. Find the Taylor series of $y = f(x)$ about x_0 in CLOSED form.

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{(n+1)}{3^{n+2}}(x-1)^n$$

e. Find the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

$$c_n = \frac{n+1}{3^{n+2}} \quad \text{or} \quad (n+1)\,3^{-(n+2)}$$

f. Find the interval I of convergence of the Taylor series $y = f(x)$ about x_0. Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

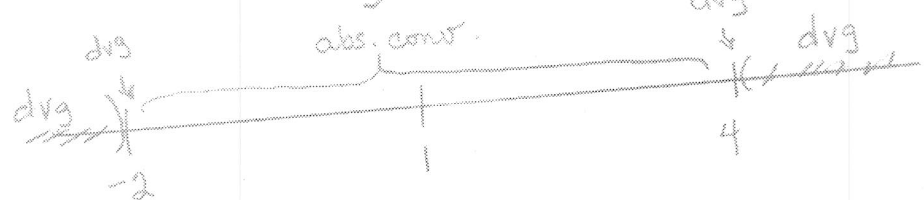
$$I = (-2, 4)$$

Ratio Test

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)\,(x-1)^{n+1}}{3^{n+3}} \cdot \frac{3^{n+2}}{(n+1)(x-1)^n}\right|$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1}\,\frac{3^{n+2}}{3^{n+3}}\left|\frac{(x-1)^{n+1}}{(x-1)^n}\right| = \frac{|x-1|}{3}\lim_{n\to\infty}\frac{n+2}{n+1}$$

$$= \frac{|x-1|}{3}\cdot 1 = \frac{|x-1|}{3} < 1 \iff |x-1| < 3$$



Check endpts

$$x = 4 \Rightarrow \sum \frac{n+1}{3^{n+2}}(x-1)^n = \sum \frac{n+1}{3^{n+2}}(4-1)^n = \sum \frac{n+1}{3^{n+2}}\cdot 3^n = \frac{1}{3^2}\sum(n+1)$$

dvg by n^{th} term test for dvg

$$x = -2 \Rightarrow \sum \frac{n+1}{3^{n+2}}(x-1)^n = \sum \frac{n+1}{3^{n+2}}(-3)^n = \sum \frac{n+1}{3^{n+2}}(-1)^n\cdot 3^n \iff$$

$$\iff \frac{1}{3^2}\sum(-1)^n(n+1). \leftarrow \text{dvg by } n^{\text{th}} \text{ term test for dvg}$$

$J = (0,2) \leftarrow$ given on page 2.

g. Consider the given interval J and fix an $N \in \mathbb{N}$. Find a good upper bound for the maximum of $|f^{(N+1)}(c)|$ on the interval J. Your answer can have an N in it but it cannot have an: x, x_0, c. (Note that J is a subset of I but Prof. G. might have picked a smaller J than I to make the problem easier.)

$$\boxed{\max_{c \in J} \left|f^{(N+1)}(c)\right| \leq \frac{(N+2)!}{2^{N+3}}}$$

From (a) we see $f^{N}(x) = \dfrac{(N+1)!}{(4-x)^{N+2}}$ so

$$|f^{(N+1)}(c)| = \frac{(N+2)!}{|4-c|^{N+3}}$$

$x \in J$ and c is btw. x & $x_0 := 1$ $\Rightarrow 0 \leq c \leq 2$

$$\Leftrightarrow -2 \leq -c \leq 0 \Rightarrow 2 \leq 4-c \leq 4 \Rightarrow 2 \leq |4-c| \leq 4 \text{ so}$$

$$|f^{(N+1)}(c)| = \frac{(N+2)!}{|4-c|^{N+3}} \leq \frac{(N+2)!}{2^{N+3}}$$

h. Consider the given interval J and fix an $N \in \mathbb{N}$. For each $x \in J$, find a good upper bound for the maximum of $|R_N(x)|$. Your answer can have an N and x in it but it cannot have an: x_0, c.

$$\boxed{|R_N(x)| \leq \frac{(N+2)\,|x-1|^{N+1}}{2^{N+3}}}$$

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-1)^{N+1} \quad \text{for some } c \text{ btw } x \text{ and } x_0 := 1.$$

$$\Rightarrow |R_N(x)| = |f^{(N+1)}(c)| \cdot \frac{1}{(N+1)!}\,|x-1|^{N+1}$$

$$\overset{g}{\leq} \frac{(N+2)!}{2^{N+3}} \cdot \frac{1}{(N+1)!}\,|x-1|^{N+1} = \frac{(N+2)\,|x-1|^{N+1}}{2^{N+3}}$$

i. **Carefully** show that $f(x) = P_\infty(x)$ for each x in the given interval J by using part (h) and showing that $\lim_{N\to\infty} |R_N(x)| = 0$ for each $x \in J$.

Fix $x \in J = (0, 2)$.

So $0 < x < 2$ and so $-1 < x-1 < 1$ so $|x-1| \leq 1$.

So

$$0 \leq |R_N(x)| \overset{(h)}{\leq} \frac{(N+2)\,|x-1|^{N+1}}{2^{N+3}} \leq \frac{(N+2)\,1^{N+1}}{2^{N+3}}$$

$$= \frac{N+2}{2^{N+3}} \leq \frac{(1.5)^{N+2}}{2^{N+3}} = \frac{(1.5)^2}{2^3}\left(\frac{1.5}{2}\right)^N \xrightarrow{N\to\infty} 0$$

N big enough
"helpful intuition"

$$\boxed{\text{b/c } \lim_{N\to\infty} r^N = 0 \text{ when } |r|<1 \text{ \& here } r = \frac{1.5}{2}}$$

j. Using a known Taylor Series (as from the handout `Commonly Used Taylor Series`) and methods from Section 11.9, find a power series expansion (in CLOSED form) for $y = f(x)$ about x_0. Also, say when this power series expansion is valid by examining when the `Commonly Used Taylor Series` is valid. **Show all your work and work in a logical fashion.**

$-2 \leftarrow 1 \rightarrow 4$

$$f(x) = \sum_{n=0}^{\infty} \boxed{\frac{n+1}{3^{n+2}}} (x-1)^n \quad \text{which is valid for} \quad \boxed{\begin{array}{l} |x-1| < 3 \\ \text{i.e. } -2 < x < 3 \end{array}}.$$

Warning: the above sum starts with $n = 0$. Hints:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} = (1-r)^{-1} \quad \text{valid when} \quad |r| < 1 \tag{1}$$

$$\frac{1}{4-x} = \frac{1}{3-(x-1)} = \frac{1}{3}\left[\frac{1}{1-\left(\frac{x-1}{3}\right)}\right] \tag{2}$$

$$D_x \frac{1}{4-x} = D_x(4-x)^{-1} = (4-x)^{-2} \tag{3}$$

$$(1)+(2) \Rightarrow \frac{1}{(4-x)} \overset{(2)}{=} \frac{1}{3}\left[\frac{1}{1-\left(\frac{x-1}{3}\right)}\right] = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x-1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x-1)^n$$

valid when $\left|\frac{x-1}{3}\right| < 1 \iff |x-1| < 3$

$$(4-x)^{-2} \overset{(3)}{=} D_x \frac{1}{4-x} \overset{\text{above}}{=} D_x\left[\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x-1)^n\right]$$

for $-2 < x < 3$

$$= \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} D_x (x-1)^n = \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} D_x (x-1)^n$$

why?

$$= \sum_{n=1}^{\infty} \frac{n}{3^{n+1}} (x-1)^{n-1} = \frac{1}{3^2} + \frac{2}{3^3}(x-1) + \frac{3}{3^4}(x-1)^2 + \frac{4}{3^5}(x-1)^3 + \cdots$$

let $n = k+1$

$$\text{or} = \sum_{k=0}^{\infty} \frac{(k+1)}{3^{k+2}} (x-1)^k$$

$\boxed{\text{now let } n = k}$

compare with your sol'n to d.