Prof. Girardi | Math 142 | Spring 2010 | 03.18.10 | Exam 2

MARK BOX		
PROBLEM	POINTS	
1 a – j	30	30
2	5	5
3	10	10
4	10	10
5 ab	10	10
6	10	10
7	10	10
8	5	5
9	10	10
%	100	100

NAME: [redacted] Sol'n

PIN: [redacted]

100 [redacted] yes!

INSTRUCTIONS:

(1) To receive credit you must:
   (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
   (b) if a line/box is provided, then:
   — show you work BELOW the line/box
   — put your answer on/in the line/box
   (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Stewart, 6^{th} ed., ET): 11.2–11.8 .

Problem Inspiration: Mostly homework and old exam problems. See the solution key for details.

1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1a. n^{th}**-term test** for an arbitrary series $\sum a_n$.
If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges.

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ < 1
- diverges if and only if $|r|$ ≥ 1

1c. p**-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p > 1
- diverges if and only if p ≤ 1

1d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
Let $f\colon [1, \infty) \to \mathbb{R}$ be so that

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a continuous function
- f is a decreasing function .

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

1e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.
If 0 $< L <$ ∞, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

1g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If ρ < 1 then $\sum a_n$ converges.
- If ρ > 1 then $\sum a_n$ diverges.
- If ρ $= 1$ then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.
If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

1i. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

1j. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=17}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

Does $\sum |a_n|$ converge? Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$ Does $\lim_{n\to\infty} |a_n| = 0$?

if NO $\Rightarrow$ $\sum a_n$ is divergent

if YES $\Downarrow$ (from first box): $\sum a_n$ is absolutely convergent

if YES $\Downarrow$ (from second box): Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is conditionaly convergent

2. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

T (F) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges

(T) F If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(T) F If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.

T (F) If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.

(T) F If $S_N = \sum_{n=1}^{N} r^n$, then $S_N = \dfrac{r - r^{N+1}}{1-r}$.

$$S_N = r + r^2 + \dots + r^N$$
$$-\ rS_N = \quad r^2 + \dots + r^N + r^{N+1}$$
$$(1-r)S_N = r - r^{N+1}$$
$$S_N = \frac{r - r^{N+1}}{1-r}$$

3. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=17}^{\infty} \frac{(-1)^n}{n}$$

- [ ] ~~absolutely~~ convergent
- [x] conditionally convergent
- [ ] divergent

Check for A.C.

$\sum_{n=17}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ p series with $p=1$ so diverges (harmonic series)

AST

$u_n = \frac{1}{n} > \frac{1}{n+1} = u_{n+1}$ ✓

$\lim_{n\to\infty} \frac{1}{n} = 0$ so by AST $\sum \frac{(-1)^n}{n}$ converges

4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n\,(n+1)\,(n+2)}}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}} = \sum \frac{1}{(n(n+1)(n+2))^{1/2}} = a_n$$

LCT

$$b_n = \frac{1}{(n^3)^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3)^{1/2}}{[n(n+1)(n+2)]^{1/2}} = 1$$ so $0 < L = 1 < \infty$ so $\sum a_n$ and $\sum b_n$ do the same thing. $\sum b_n = \sum \frac{1}{(n^3)^{1/2}} = \sum \frac{1}{n^{3/2}}$ p-series with $p = \frac{3}{2} > 1$ so conv. since $\sum b_n$ converges the $\sum \frac{1}{\sqrt{n(n+1)(n+2)}}$ converges by the LCT.

5. Let

$$a_n = \frac{n!}{(2n-1)!}$$

5a. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{n+1}{2n(2n+1)}} \checkmark$$

5b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n-1)!}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{[2(n+1)-1]!} \cdot \frac{(2n-1)!}{n!} = \frac{n!(n+1)}{(2n+1)!} \cdot \frac{(2n-1)!}{n!}$$

$$= \frac{\cancel{n!}(n+1)}{\cancel{(2n-1)!}2n(2n+1)} \cdot \frac{\cancel{(2n-1)!}}{\cancel{n!}} = \frac{n+1}{2n(2n+1)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2n^2+2n} = 0 = \rho < 1$$ so by the ratio test

$\sum \frac{n!}{(2n-1)!}$ converges

nice

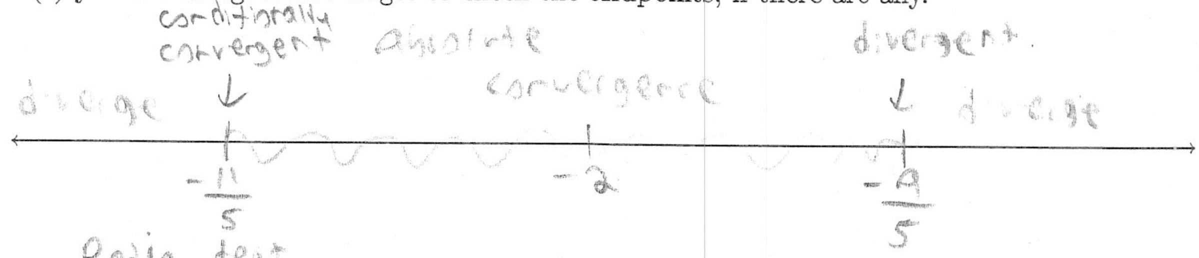
6. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(5x+10)^n}{n}$$

Hint: $(5x+10)^n = [5(x+2)]^n = 5^n(x+2)^n = 5^n(x-(-2))^n$

The center is $x_0 =$ -2 and the radius of convergence is $R =$ $\frac{1}{5}$.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



Ratio test

$$\lim_{n\to\infty}\left|\frac{5^{n+1}(x+2)^{n+1}}{(n+1)}\cdot\frac{n}{5^n(x+2)^n}\right| = \lim_{n\to\infty}\left|\frac{5(x+2)n}{n+1}\right| = |5(x+2)|\lim_{n\to\infty}\frac{n}{n+1} =$$

$\rightarrow 5|x+2|$

$5|x+2|<1 = |x+2| < \frac{1}{5}$

Test endpoints

$x = -\frac{9}{5}$ $\sum \frac{(1)^n}{n} = \sum \frac{1}{n}$ p-series with $p=1$ so dvg (harmonic)

$x = -\frac{11}{5}$ $\sum \frac{(-1)^n}{n}$

check for A.C.

nice

$\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ dvg

AST

$u_n = \frac{1}{n} > \frac{1}{n+1} = u_{n+1}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ so by AST $\sum \frac{(-1)^n}{n}$ convg

7. Consider the formal power series

$$\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}.$$

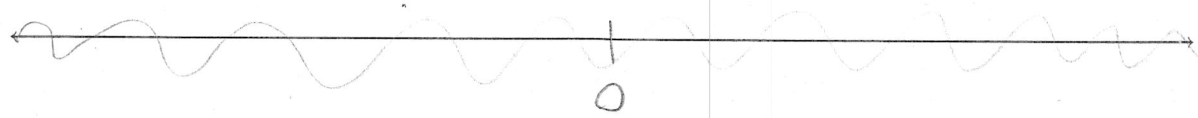
Hint 1: $\frac{x^n}{(\ln n)^n} = \left[\frac{x}{\ln n}\right]^n$ so would you rather use the root test or the ratio test?

Hint 2: $\ln(a^r) = r\ln(a)$ but $(\ln(a))^r \neq r\ln(a) +$

The center is $x_0 =$ 0 and the radius of convergence is $R =$ ∞.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.

absolutely convergent for all X



Root test:

$$\lim_{n\to\infty}\left[\left(\frac{X}{\ln n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{X}{\ln n} = X\lim_{n\to\infty}\frac{1}{\ln n} = X\cdot 0 = 0$$

so convergent for all X

8. Fill-in the 6 blanks.

Consider the power series

$$\sum_{n=1}^{\infty} (-1)^n \, a_n x^n$$

where all of the a_n's are positive. Let's say that you know that

if	$0 < x < 17$	then $\sum (-1)^n \, a_n x^n$ converges
if	$x = 17$	then $\sum (-1)^n \, a_n x^n$ conditionally converges
if	$17 < x$	then $\sum (-1)^n \, a_n x^n$ diverges .

Then this power series has:

center at $x_0 =$ __0__ and radius of convergence $R =$ __17__ .

Also, what can you say about the following interval? Fill in the blanks below with:

- is absolutely convergent
- is conditionally convergent
- is divergent
- inconclusive (not enough information given to decide in general).

if	$-17 < x < 0$	then $\sum (-1)^n \, a_n x^n$	absolutely convergent
if	$x < -17$	then $\sum (-1)^n \, a_n x^n$	divergent
if	$x = 0$	then $\sum (-1)^n \, a_n x^n$	absolutely convergent
if	$x = -17$	then $\sum (-1)^n \, a_n x^n$	divergent

$\sum (-1)^n a_n (17)^n$ this conditionally converget

$\sum (-1)^n a_n (-17)^n$

drg bc

9. Geometric Series. Let, for $N \geq 102$,

$$s_N = \sum_{n=102}^{N} 3\frac{2^n}{7^{n+1}}.$$

9a. Do some algebra to write s_N as $\sum_{n=102}^{N} c\, r^n$ for an appropriate constant c and ratio r.

$$\boxed{s_N = \sum_{n=102}^{N} \frac{3}{7}\left(\frac{2}{7}\right)^n}$$

$$3\sum\frac{2^n}{7^n 7} = \frac{3}{7}\sum\frac{2^n}{7^n} = \sum\frac{3}{7}\left(\frac{2}{7}\right)^n$$

9b. Using the method from class (rather than some formula), find an expression for s_N in closed form (i.e. without a summation $\sum$ sign nor some dots $\ldots$.).

$$\boxed{s_N = \frac{3}{5}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right]}$$

$$s_N = \frac{3}{7}\left(\frac{2}{7}\right)^{102} + \cancel{\frac{3}{7}\left(\frac{2}{7}\right)^{103}} + \ldots + \cancel{\frac{3}{7}\left(\frac{2}{7}\right)^{N}}$$

$$-\left(\frac{2}{7}s_N = \qquad + \cancel{\frac{3}{7}\left(\frac{2}{7}\right)^{103}} + \ldots + \cancel{\frac{3}{7}\left(\frac{2}{7}\right)^{N}} + \frac{3}{7}\left(\frac{2}{7}\right)^{N+1}\right)$$

$$\frac{5}{7}s_N = \frac{3}{7}\left(\frac{2}{7}\right)^{102} - \frac{3}{7}\left(\frac{2}{7}\right)^{N+1}$$

$$\frac{5}{7}s_N = \frac{3}{7}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right]$$

$$s_N = \frac{3}{5}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right]$$

nice

9c. Does $\sum_{n=102}^{\infty} 3\frac{2^n}{7^{n+1}}$ converge or diverge? If it converges, find its sum. Justify your answer.

$\sum_{n=102}^{\infty} 3\frac{2^n}{7^{n+1}}$ converges to $\frac{3}{5}\left(\frac{2}{7}\right)^{102}$

$\sum 3\frac{2^n}{7^{n+1}} = \sum\frac{3}{7}\left(\frac{2}{7}\right)^n$ geometric series with $|r| = \frac{2}{7} < 1$ so $\sum 3\frac{2^n}{7^{n+1}}$ conv

$$s_N = \frac{3}{5}\left[\left(\frac{2}{7}\right)^{102} - \left(\frac{2}{7}\right)^{N+1}\right]$$

$$\lim_{N\to\infty} s_N = \frac{3}{5}\left(\frac{2}{7}\right)^{102}$$