$\boxed{1}$ Find and simplify if necessary

(a) $D_x\left[e^{3x^2+1}\right] = 6xe^{3x^2+1}$ since $(e^u)' = u'e^u$.

(b) $D_x\left[\ln(3x^2+17)\right] = \frac{6x}{3x^2+17}$ since $(\ln(u))' = \frac{u'}{u}$.

(c) $D_x\left[(1+x)^{2x}\right]$.

let $y = (1+x)^{2x}$, then $\ln y = \ln(1+x)^{2x}$ or

$\ln y = 2x\ln(1+x)$. Differentiate both sides with respect to $x$ gives

$$\frac{y'}{y} = 2x \cdot \frac{1}{1+x} + 2\ln(1+x)$$

$$\Leftrightarrow y' = y\left[\frac{2x}{1+x} + 2\ln(1+x)\right] = (1+x)^{2x}\left[\frac{2x}{1+x} + 2\ln(1+x)\right)$$

$$\Rightarrow D_x\left[(1+x)^{2x}\right] = (1+x)^{2x}\left[\frac{2x}{1+x} + 2\ln(1+x)\right].$$

(d) $D_x\left[\sin^3(4x)\right] = 3\sin^2(4x)\cdot(\sin(4x))'$

$= 3\sin^2(4x)\cos(4x)\cdot 4$

$= 12\sin^2(4x)\cos(4x)$.

(e) $\frac{d}{dx}e^{\tan x} = (\tan x)'e^{\tan x} = \sec^2 x e^{\tan x}$.

(f) $\frac{d}{dx}\left[\ln x\right]^{2x+3}$

let $y = (\ln x)^{2x+3} \Leftrightarrow \ln y = \ln(\ln x)^{2x+3}$

$\Leftrightarrow \ln y = (2x+3)\ln(\ln x)$. Differentiate both sides

$$\frac{y'}{y} = (2x+3)\left[\frac{(\ln x)'}{\ln x}\right] + 2\ln(\ln x) = (2x+3)\left(\frac{1}{x\ln x}\right) + 2\ln(\ln x)$$

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$$\Rightarrow y' = y\left[\frac{2x+3}{x\ln x} + 2\ln(\ln x)\right]$$

So, $$\frac{d}{dx}[\ln x]^{2x+3} = (\ln x)^{2x+3}\left[\frac{2x+3}{x\ln x} + 2\ln(\ln x)\right]$$

(g) $$D_x\left[17^{3x^2+1}\right] = 6x \cdot 17^{3x^2+1}\ln(17)$$

(h) $$D_x[\ln(\cos(4x))]$$
$$= \frac{(\cos 4x)'}{\cos 4x} = \frac{-4\sin 4x}{\cos 4x}.$$

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$\boxed{2}$ Integrate each of the following using the appropriate method.

(a) $\int \ln x \, dx$. This was done in class.

(b) $\int \sin^2 x \, dx$. Done in class.

(c) $\int \sin^3 x \, dx$. same as above.

(d) $\int x^2 \sin x \, dx$

let $u = x^2 \Rightarrow du = 2x\,dx$ and $dv = \sin x\,dx \Updownarrow v = -\cos x$

$$= -x^2\cos x + 2\int x\cos x\,dx \text{ - - - - } ①$$

we need to do by part again for $\int x\cos x\,dx$

Let $u = x \Leftrightarrow du = dx$ and $dv = \cos x dx$

$\Leftrightarrow v = \sin x$

$$\text{So, } \int x \cos x dx = x \sin x - \int \sin x dx$$
$$= x \sin x + \cos x + C_1$$

$$\text{Hence, } \int x^2 \sin x dx = -x^2 \cos x + 2\int x \cos x dx$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + 2C_1$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

(e) $\displaystyle \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

Since the degree of numerater is greater than that of the denombnator, so we perform a long division.

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ \hline & x^3-x^2+3x+1 \\ & x^3-x^2-x+1 \\ \hline & 4x \end{array}$$

$$= \int \left( x+1 + \frac{4x}{x^3-x^2-x+1} \right) dx$$

Notice that $x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x-1)(x^2-1)$

$= (x-1)(x-1)(x+1) = (x+1)(x-1)^2$

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

- When $x=1$, $4 = 2C \Rightarrow C = 2$
- when $x=-1$, $-4 = 4A$ [illegible] $\Rightarrow A =$ [illegible] $-1$
- when $x=0$, $0 = A - B + C \Rightarrow B =$ [illegible] $1$

So, $\int \left(x+1 + \frac{\cancel{2}\,1}{x-1} + \frac{2}{(x-1)^2}\right) dx + \int \frac{-1}{x+1}\,dx$

$= \int x\,dx + \int 1\,dx + \overset{1}{\cancel{2}} \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x+1}$

$= \frac{x^2}{2} + x + \cancel{2}\ln|x-1| - \frac{2}{x-1} + K \cdot - \ln|x+1|$

(f) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$. This can be done by u-substitution. let $u = 1-x^2 \Leftrightarrow du = -2x\,dx$

$\Updownarrow$

$x^2 = 1-u$

$\underset{\substack{\uparrow \\ \text{rewrite it as}}}{=} \int \frac{x^2 \cdot x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{x^2 \cdot -2x\,dx}{\sqrt{1-x^2}}$

$= -\frac{1}{2}\int \frac{(1-u)\,du}{\sqrt{u}} = -\frac{1}{2}\int (1-u)u^{-1/2}\,du$

$= -\frac{1}{2}\int (u^{-1/2} - u^{1/2})\,du$

$$= -\frac{1}{2}\int u^{-1/2}\,du + \frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + 1/2\,\frac{u^{3/2}}{3/2} + t$$

$$= -\sqrt{u} + \frac{1}{3}u\sqrt{u} + t$$

$$= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)\sqrt{1-x^2} + t.$$

(g) $\int x^2 \arctan x\,dx$. We can use the trick

~~[illegible]~~

This tells us to choose $u = \arctan x \iff du = \frac{dx}{1+x^2}$

and $dv = x^2\,dx \iff v = \frac{x^3}{3}$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int\frac{x^3\,dx}{1+x^2}$$

$\hookrightarrow$ long division

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$\begin{array}{rl} & x \\ 1+x^2\,\big) & x^3 \\ & x^3+x \\ \hline & -x \end{array}$$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int\frac{x\,dx}{1+x^2}$$

$\hookrightarrow$ $u = 1+x^2$, $du = 2x\,dx$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\frac{x^2}{2} + \frac{1}{6}\ln(1+x^2) + C.$$

$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6} + \frac{\ln(1+x^2)}{6} + C.$$

(h) $\int e^x \cos x \, dx$

This is Example 5 on page 516.

(i) $\displaystyle\int \frac{x}{x^4+4x^2+8}\,dx$

rewrote 8 as 4 + 4

$$x^4+4x^2+8 = \underbrace{x^4+4x^2+4}+4$$

by completing the square method

$$= (x^2+2)^2+4 = 4+(x^2+2)^2$$

$$= 4\left[1+\left(\frac{x^2+2}{2}\right)^2\right].$$

$$\text{let } \frac{x^2+2}{2} = u \Leftrightarrow \frac{2x}{2}dx = du \Leftrightarrow x\,dx = du$$

$$= \int \frac{du}{4[1+u^2]} = \frac{1}{4}\int\frac{du}{1+u^2} = \frac{1}{4}\text{Arctan}(u)+C$$

$$= \frac{1}{4}\text{Arctan}\left(\frac{x^2+2}{2}\right)+C.$$

(j)

$$\frac{x^4+2x+2}{x^5+x^4} = \frac{x^4+2x+2}{x^4(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}+\frac{E}{x+1} = \frac{Ax^3(x+1)+Bx^2(x+1)+Cx^1(x+1)+D(x+1)+Ex^4}{x^4(x+1)}$$

$$\Rightarrow x^4+2x+2 = Ax^3(x+1)+Bx^2(x+1)+Cx(x+1)+D(x+1)+Ex^4$$

$\boxed{x=0}$ $\boxed{2 = D}$

$x=-1$ $1-2+2 = E \Rightarrow \boxed{E=1}$

$x^4$ $1 = A+E \Longrightarrow A = 1-E = 1-1 = 0$

$x^3$ $0 = A+B \Longrightarrow B = -A = 0$

$x^2$ $0 = B+C \Longrightarrow C = -B = 0$

$x^1$ $2 = C+D$

Constant $2 = D$

$$\int\frac{x^4+2x+2}{x^5+x^4}dx = \int\left[\frac{2}{x^4}+\frac{1}{x+1}\right]dx = \boxed{\frac{-2x^{-3}}{-3}+\ln|x+1|+C}$$

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(k) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

We will do it by trig-substitution.

$x = 2\sin\theta$ where $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

then $4-x^2 = 4-4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta$

$dx = 2\cos\theta\,d\theta$

$$= \int \frac{4\sin^2\theta \cdot \cancel{2\cos\theta}\,d\theta}{\cancel{2\cos\theta}}$$

$$= 4\int \sin^2\theta\,d\theta = 4\int \frac{1-\cos 2\theta}{2}\,d\theta$$

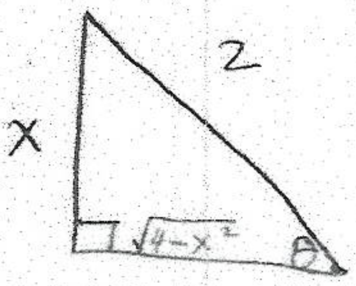
$$= 2\int d\theta - 2\int \cos 2\theta\,d\theta = 2\theta - \frac{2}{2}\sin(2\theta) + C$$

$$= 2\theta - \sin(2\theta) + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \cancel{2}\left(\frac{x}{\cancel{2}}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4-x^2} + C.$$



$x = 2\sin\theta \rightarrow \sin\theta = \frac{x}{2}$

$\Updownarrow$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

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(L) $$\int_0^\infty \frac{dx}{1+x} = \lim_{t\to\infty}\int_0^t \frac{dx}{1+x}$$

$$= \lim_{t\to\infty} \ln|1+x|\Big|_0^t = \lim_{t\to\infty}\left(\ln|1+t| - \overset{0}{\cancel{\ln|1|}}\right)$$

$$= \ln(\infty) = \infty .$$

(m) $$\int_{-\infty}^{\infty} \frac{x}{x^2+1}dx = \int_{-\infty}^{0} \frac{x}{x^2+1}dx + \int_0^\infty \frac{x\,dx}{x^2+1}$$

$$= \lim_{a\to-\infty}\int_a^0 \frac{x\,dx}{x^2+1} + \lim_{b\to\infty}\int_0^b \frac{x\,dx}{x^2+1}$$

$$= \lim_{a\to-\infty} \frac{1}{2}\ln(x^2+1)\Big|_a^0 + \lim_{b\to\infty}\frac{1}{2}\ln|x^2+1|\Big|_0^b$$

$$= \lim_{a\to-\infty}\left(\overset{0}{\cancel{\tfrac{1}{2}\ln(1)}} - \tfrac{1}{2}\ln(a^2+1)\right) +$$
$$\lim_{b\to\infty}\left(\tfrac{1}{2}\ln(b^2+1) - \overset{0}{\cancel{\tfrac{1}{2}\ln(1)}}\right)$$

$$= -\frac{1}{2}\ln(\infty) + \frac{1}{2}\ln(\infty) = \infty .$$

let $u = x^2+1$
$du = 2x\,dx$

$$\frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\int\frac{du}{u}$$
$$= \frac{1}{2}\ln|u| + C$$

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$\boxed{3}$ Find the limit.

(a) $\lim\limits_{x\to\infty} x^{1/x} \longrightarrow \infty^{\frac{1}{\infty}} = \infty^{0}$ indeterminate

let $y = x^{1/x} \iff \ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$

So, $\lim\limits_{x\to\infty} \ln(y) = \lim\limits_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$

$$\overset{\text{l'Hopital's Rule}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \left(\frac{1}{x}\right) = 0.$$

So, $\ln(y) = 0 \Longrightarrow y = e^{0} = 1.$

Hence $\lim\limits_{x\to\infty} x^{1/x} = 1.$

(b) $\lim\limits_{n\to\infty} \frac{12n^{17}+188n^{7}-19n}{4n^{18}-n^{9}+10} = \lim\limits_{n\to\infty} \frac{\frac{12}{n}+\frac{188}{n^{9}}-\frac{19}{n^{17}}}{4-\frac{1}{n^{9}}+\frac{10}{n^{18}}}$

$$= \frac{0}{4} = 0.$$

(c) $\lim\limits_{x\to\infty} \left(1+\frac{c}{x}\right)^{x}, \ c \neq 0$

$\lim\limits_{x\to\infty} \left(1+\frac{c}{x}\right)^x \longleftarrow 1^\infty$ indeterminate form. See the L'Hopital handout on how to deal w/ such an indet. form.

Idea:

$$\ln\left(1+\frac{c}{x}\right)^x \xrightarrow{x\to\infty} \square$$

so $\left(1+\frac{c}{x}\right)^x = e^{\ln\left(1+\frac{c}{x}\right)^x} \longrightarrow e^{\square}$.

---

$$\lim_{x\to\infty} \ln\left(1+\frac{c}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{c}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+cx^{-1})}{x^{-1}}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty}\frac{\frac{1}{(1+cx)^{-1}}\cdot c(-1x^{-2})}{(-1x^{-2})} = \lim_{x\to\infty}\frac{c}{1+cx} = \frac{c}{1+0} = c$$

so $\underset{\sim}{\ln}\left(1+\frac{c}{x}\right)^x \xrightarrow{x\to\infty} c$ so $\left(1+\frac{c}{x}\right)^x \xrightarrow{x\to\infty} \boxed{e^c}$

**3d** $\lim\limits_{n\to\infty}\frac{n^{17,000}}{e^n}$ ... thinkin' .... $e^n$ grows so must faster than $n^{17,000}$ so it should be zero.

**WAY #1** $\lim\limits_{n\to\infty}\frac{n^{17,000}}{e^n} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{n\to\infty}\frac{(17,000)!}{e^n} = \boxed{0}$

↑ do L'Hopital 17,000 times to get

**WAY #2** $0 \le \frac{n^{17,000}}{e^n} \le \frac{n^{17,000}}{n^{17,001}} = \frac{1}{n} \to 0.$

↑ for n big enough $n^{17,001} \le e^n$

so by squeeze theorem, $\lim\limits_{n\to\infty}\frac{n^{17,000}}{e^n} = 0$.

4. $S_N = \sum_{n=5}^{N} \frac{8(3^n)}{4^{n+2}} = \sum_{n=5}^{N} \frac{2\cdot 4}{4\cdot 4}\;\frac{3^n}{4^n} = \sum_{n=5}^{N} \frac{1}{2}\left(\frac{3}{4}\right)^n$

$$
\begin{array}{rl}
S_N = & \frac{1}{2}\left(\frac{3}{4}\right)^5 + \frac{1}{2}\left(\frac{3}{4}\right)^6 + \frac{1}{2}\left(\frac{3}{4}\right)^7 + \dots + \frac{1}{2}\left(\frac{3}{4}\right)^N \\
\left(\frac{3}{4}\right) S_N = & \qquad\qquad\;\; \frac{1}{2}\left(\frac{3}{4}\right)^6 + \frac{1}{2}\left(\frac{3}{4}\right)^7 + \dots + \frac{1}{2}\left(\frac{3}{4}\right)^N + \frac{1}{2}\left(\frac{3}{4}\right)^{N+1} \\
\hline
\end{array}
$$

$$S_N - \frac{3}{4}S_N = \frac{1}{2}\left(\frac{3}{4}\right)^5 - \frac{1}{2}\left(\frac{3}{4}\right)^{N+1}$$

$$\left(1-\frac{3}{4}\right) S_N = \frac{1}{2}\left[\left(\frac{3}{4}\right)^5 - \left(\frac{3}{4}\right)^{N+1}\right]$$

$$\frac{1}{4}\; S_N = \frac{1}{2}\left[\left(\frac{3}{4}\right)^5 - \left(\frac{3}{4}\right)^{N+1}\right]$$

$$S_N = \frac{4}{2}\left[\left(\frac{3}{4}\right)^5 - \left(\frac{3}{4}\right)^{N+1}\right]$$

$$\boxed{S_N = 2\left[\left(\frac{3}{4}\right)^5 - \left(\frac{3}{4}\right)^{N+1}\right]}$$

$$\sum_{n=5}^{\infty} \frac{8(3^n)}{4^{n+2}} = \lim_{N\to\infty} \sum_{n=5}^{N} \frac{8(3^n)}{4^{n+2}} \overset{\downarrow}{=} \lim_{N\to\infty} 2\left[\left(\frac{3}{4}\right)^5 - \left(\frac{3}{4}\right)^{N+1}\right]$$

$$= 2\left[\left(\frac{3}{4}\right)^5 - 0\right] \quad \longleftarrow \quad \boxed{\begin{array}{l}\text{if } |r|<1 \\ \text{then } \lim_{n\to\infty} r^n = 0\end{array}}$$

$$= \boxed{2\left(\frac{3}{4}\right)^5}$$

$\boxed{5}$ Decide if the given series diverges, converges conditionally, or converges absolutely.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

let $a_n = \frac{1}{\sqrt{n}} \Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and

$a_{n+1} = \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$ $\forall n \geq 1$.

So, the two conditions for AST are satisfied.

Hence it converges.

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which

diverges by p- Test $(p = 1/2 < 1)$.

Therefore, the given series converges conditionally.

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{(3^n)n!}{(2n)!}$

let $u_n = \frac{(3^n)n!}{(2n)!} \Rightarrow u_{n+1} = \frac{(3^n \cdot 3^1)(n+1)!}{(2n+2)!}$

So, $\rho = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{3^n \cdot 3^1 (n+1) n!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{3^n \cdot (n)!} \right|$

$= \lim_{n\to\infty} \left| \frac{3(n+1)}{2(n+1)(2n+1)} \right| = \lim_{n\to\infty} \left| \frac{3}{2(2n+1)} \right| = 0 < 1$

So, by the ratio test for absolute convergence, the given series converges absolutely.

(c) $\sum_{n=7}^{\infty} (-1)^n \frac{n}{1+n^2}$ , let $a_n = \frac{n}{1+n^2}$

$\lim_{n \to \infty} a_n = 0$ and let $f(x) = \frac{x}{1+x^2}$

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} \leq 0 \ \forall \ x \geq 7.$$

Hence by AST, it converges.

$$\sum_{n=7}^{\infty} \left| (-1)^n \frac{n}{1+n^2} \right| = \sum_{n=7}^{\infty} \frac{n}{1+n^2}$$

Since we showed $f'(x) < 0 \ \forall \ x \geq 7$, the integral test is applicable.

$$\int_7^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_7^b \frac{x}{1+x^2}\,dx = \frac{1}{2} \lim_{b \to \infty} \int_7^b \frac{2x\,dx}{1+x^2}$$

$$= \frac{1}{2} \lim_{b \to \infty} \ln(1+x^2) \Big|_7^b$$

$$= \frac{1}{2} \lim_{b \to \infty} \left[ \ln(1+b^2) - \ln(50) \right]$$

$$= \frac{1}{2} \left[ \infty - \ln(50) \right] = \infty.$$ So, it diverges.

Hences the given series converges conditionally.

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(d) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n}=\frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which converges since $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges. The series is conditionally convergent since $\frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

(e) $\lim_{n\to\infty}\frac{n}{10n+1}=\frac{1}{10}\neq 0$. Thus the sequence of partial sums does not converge; the series diverges.

(f) $\lim_{n\to\infty}\frac{1}{n\ln n}=0;\ \frac{1}{n\ln n}>\frac{1}{(n+1)\ln(n+1)}$ is equivalent to $(n+1)^{n+1}>n^n$ which is true for all $n>0$ so $a_n>a_{n+1}$. The alternating series converges.

$\sum_{n=2}^{\infty}|u_n|=\sum_{n=2}^{\infty}\frac{1}{n\ln n};\ \frac{1}{x\ln x}$ is continuous, positive, and nonincreasing on $[2,\infty)$.

Using $u=\ln x,\ du=\frac{1}{x}dx$,

$\int_{2}^{\infty}\frac{1}{x\ln x}dx=\int_{\ln 2}^{\infty}\frac{1}{u}du=\left[\ln|u|\right]_{\ln 2}^{\infty}=\infty$. Thus, $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges and $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln n}$ is conditionally convergent.

(g) $\frac{|u_{n+1}|}{|u_n|}=\frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}}=\cancel{\frac{n^4}{2(n+1)^4}};\ =\frac{(n+1)^4}{2n^4}$

$\lim_{n\to\infty}\frac{n^4}{2(n+1)^4}=\frac{1}{2}<1$.

The series is absolutely convergent.

(h) $a_n=\frac{n}{n^2+1};\ \frac{n}{n^2+1}>\frac{n+1}{(n+1)^2+1}$ is equivalent to $n^2+n-1>0$, which is true for $n>1$, so $a_n>a_{n+1};\ \lim_{n\to\infty}\frac{n}{n^2+1}=0$, hence the alternating series converges. Let $b_n=\frac{1}{n}$, then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=1;0<1<\infty$. Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ also diverges. The series is conditionally convergent.

(i) $\cos n\pi=(-1)^n=\frac{1}{(-1)}(-1)^{n+1}$ so the series is

$-1\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, $-1$ times the alternating harmonic series. The series is conditionally convergent.

(j) $|\sin n|\le 1$ for all $n$, so

$\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{|\sin n|}{n\sqrt{n}}\le\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which converges since $\frac{3}{2}>1$. Thus the series is absolutely convergent.

(k) $a_n=\frac{1}{\sqrt{n(n+1)}};\ \frac{1}{\sqrt{n(n+1)}}>\frac{1}{\sqrt{(n+1)(n+2)}}$ and $\lim_{n\to\infty}\frac{1}{\sqrt{n(n+1)}}=0$ so the alternating series converges.

Let $b_n=\frac{1}{n}$, then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1;$

$0<1<\infty$.

Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ also diverges.

The series is conditionally convergent.

(l) $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{n^2}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n^2};\ \lim_{n\to\infty}\frac{3^{n+1}}{n^2}\neq 0$, so the series diverges.

**6.** Consider the following formal power series. Make a diagram (as we did in class) indicating for which $x$'s this series is: absolutely convergent, conditionally convergent, divergent. Indicate your reasoning. Don't forget to check the endpoints.

6a Find the interval of convergence of $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots + (-1)^{n-1}\frac{1}{n}x^n \cdots$.

The ratio test yields

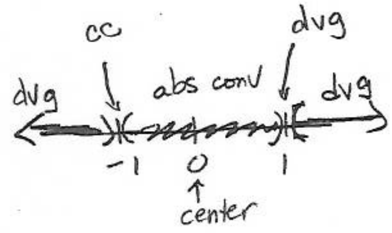
$$\lim_{n\to+\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = |x|\lim_{n\to+\infty}\frac{n}{n+1} = |x|$$

The series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Individual tests *must* be made at the endpoints $x = 1$ and $x = -1$:

For $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ and is conditionally convergent.

For $x = -1$, the series becomes $-(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots)$ and is divergent.

Thus the given series converges on the interval $-1 < x \le 1$.



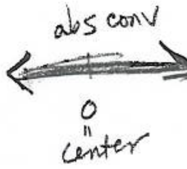
R = 1.

---

6b Find the interval of convergence of $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = |x|\lim_{n\to+\infty}\frac{1}{n+1} = 0$$

The given series converges for all values of $x$.



$R = \infty$

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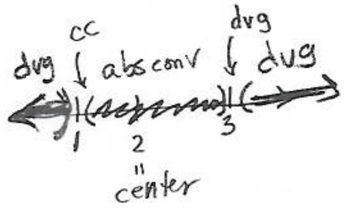
6c Find the interval of convergence of $\frac{x-2}{1} + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} + \cdots + \frac{(x-2)^n}{n} + \cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{(x-2)^{n+1}}{n+1}\frac{n}{(x-2)^n}\right| = |x-2|\lim_{n\to+\infty}\frac{n}{n+1} = |x-2|$$

The series converges absolutely for $|x-2| < 1$ or $1 < x < 3$ and diverges for $|x-2| > 1$ or for $x < 1$ and $x > 3$.

For $x = 1$ the series becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$, and for $x = 3$ it becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. The first converges, and the second diverges. Thus the given series converges on the interval $1 \le x < 3$ and diverges elsewhere.



R = 1

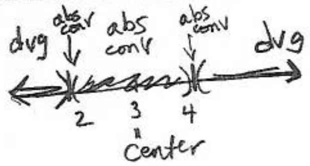
---

6d Find the interval of convergence of $1 + \frac{x-3}{1^2} + \frac{(x-3)^2}{2^2} + \frac{(x-3)^3}{3^2} + \cdots + \frac{(x-3)^{n-1}}{(n-1)^2} + \cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{(x-3)^n}{n^2}\frac{(n-1)^2}{(x-3)^{n-1}}\right| = |x-3|\lim_{n\to+\infty}\left(\frac{n-1}{n}\right)^2 = |x-3|$$

The series converges absolutely for $|x-3| < 1$ or $2 < x < 4$ and diverges for $|x-3| > 1$ or for $x < 2$ and $x > 4$.

For $x = 2$ the series becomes $1 - 1 + \frac{1}{4} - \frac{1}{9} + \cdots$, and for $x = 4$ it becomes $1 + 1 + \frac{1}{4} + \frac{1}{9} + \cdots$. Since both are absolutely convergent, the given series converges absolutely on the interval $2 \le x \le 4$ and diverges elsewhere. Note that the first term of the series is not given by the general term with $n = 0$.



R = 1

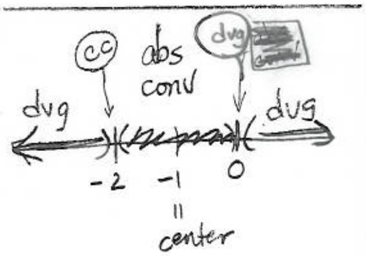
---

6e Find the interval of convergence of $\frac{x+1}{\sqrt{1}} + \frac{(x+1)^2}{\sqrt{2}} + \frac{(x+1)^3}{\sqrt{3}} + \cdots + \frac{(x+1)^n}{\sqrt{n}} + \cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{(x+1)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{(x+1)^n}\right| = |x+1|\lim_{n\to+\infty}\sqrt{\frac{n}{n+1}} = |x+1|$$

The series converges absolutely for $|x+1| < 1$ or $-2 < x < 0$ and diverges for $x < -2$ and $x > 0$.

For $x = -2$ the series becomes $-1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \cdots$, and for $x = 0$ it becomes $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$. The first is convergent, and the second is divergent (why?). Thus, the given series converges on the interval $-2 \le x < 0$ and diverges elsewhere.



R=1

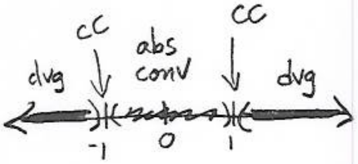
/15

.6f Find the interval of convergence of $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{x^{2n+1}}{2n+1}\,\frac{2n-1}{x^{2n-1}}\right| = x^2 \lim_{n\to+\infty}\frac{2n-1}{2n+1} = x^2$$

The series is absolutely convergent on the interval $x^2 < 1$ or $-1 < x < 1$.

For $x = -1$ the series becomes $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7}\cdots$, and for $x = 1$ it becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots$. Both series converge; thus the given series converges for $-1 \le x \le 1$ and diverges elsewhere.



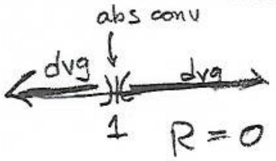
$R=1$

---

.6g Find the interval of convergence of $(x-1) + 2!(x-1)^2 + 3!(x-1)^3 + \cdots + n!(x-1)^n + \cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{(n+1)!(x-1)^{n+1}}{n!(x-1)^n}\right| = |x-1|\lim_{n\to+\infty}(n+1) = \infty$$

The series converges for $x = 1$ only.



$R=0.$

---

#7. Given $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ when $|r|<1$.

7a) $$f(t) = \frac{1}{9+t} = \frac{1}{9}\cdot\frac{1}{1-\left(-\frac{t}{9}\right)} \overset{r=-\frac{t}{9}}{=\!=\!=} \frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{t}{9}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{9}\,\frac{(-1)^n\ t^n}{9^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{9^{n+1}}\ t^n}$$

Valid $\Leftrightarrow \left|-\frac{t}{9}\right| < 1 \Leftrightarrow \boxed{|t| < 9}$

7b) $$g(x) = \frac{7}{9+4x} = 7\cdot\frac{1}{9+4(x-3)+12} = 7\cdot\frac{1}{21+4(x-3)}$$

$$= \frac{7}{21}\,\frac{1}{1-\underbrace{\left[-\frac{4}{21}(x-3)\right]}_{"r"}} = \frac{7}{21}\sum_{n=0}^{\infty}\left[-\frac{4}{21}(x-3)\right]^n$$

$$= \sum_{n=0}^{\infty}\frac{7}{21}\,\frac{(-1)^n(4)^n}{(21)^n}(x-3)^n = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 7\cdot 4^n}{(21)^{n+1}}(x-3)^n}$$

valid $\Leftrightarrow \left|-\frac{4}{21}(x-3)\right| < 1 \Leftrightarrow \boxed{|x-3| < \frac{21}{4}}$

#8 – Solutions are boxed

~~1.~~ 8a Find the volume generated by revolving the first-quadrant area bounded by the parabola $y^2 = 8x$ and its latus rectum $(x = 2)$ about the $x$ axis.

We divide the plane area vertically, as can be seen in Fig. 41-7. When the approximating rectangle is revolved about the $x$ axis, a disc whose radius is $y$, whose height is $\Delta x$, and whose volume is $\pi y^2 \Delta x$ is generated. The sum of the volumes of $n$ discs, corresponding to the $n$ approximating rectangles, is $\Sigma\, \pi y^2 \Delta x$, and the required volume is

$$V = \int_a^b dV = \int_0^2 \pi y^2\, dx = \boxed{\pi \int_0^2 8x\, dx} = 4\pi x^2 \Big|_0^2 = 16\pi \text{ cubic units}$$

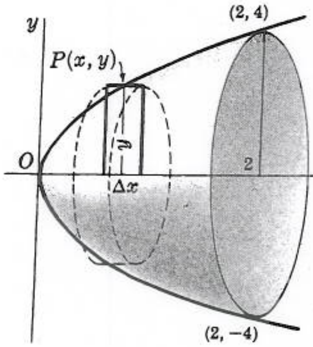


Fig. 41-7

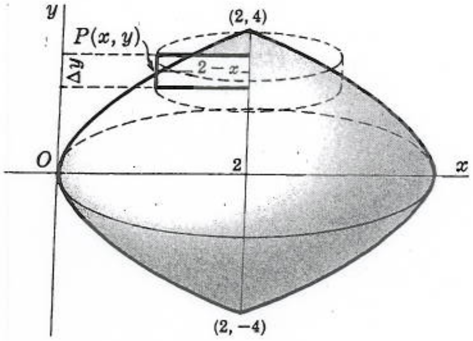


Fig. 41-8

~~2.~~ 8b Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum $(x = 2)$ about the latus rectum.

We divide the area horizontally, as can be seen in Fig. 41-8. When the approximating rectangle is revolved about the latus rectum, it generates a disc whose radius is $2 - x$, whose height is $\Delta y$, and whose volume is $\pi(2 - x)^2 \Delta y$. The required volume is then

$$V = \int_{-4}^{4} \pi(2 - x)^2\, dy = 2\pi \int_0^4 (2 - x)^2\, dy = \boxed{2\pi \int_0^4 \left(2 - \frac{y^2}{8}\right)^2 dy} = \frac{256}{15}\pi \text{ cubic units}$$

~~3.~~ 8c Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum $(x = 2)$ about the $y$ axis.

We divide the area horizontally, as shown in Fig. 41-9. When the approximating rectangle is revolved about the $y$ axis, it generates a washer whose volume is the difference between the volumes generated by revolving the rectangle $ECDF$ (of dimensions 2 by $\Delta y$) and the rectangle $EABF$ (of dimensions $x$ by $\Delta y$) about the $y$ axis, that is, $\pi(2)^2 \Delta y - \pi(x)^2 \Delta y$. The required volume is then

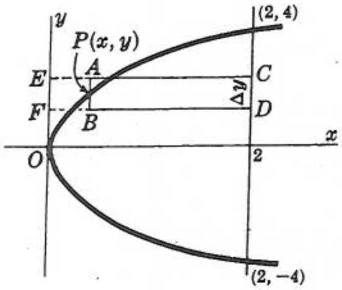


Fig. 41-9

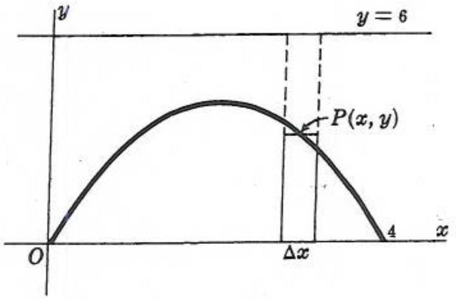


Fig. 41-10

$$V = \int_{-4}^{4} 4\pi\, dy - \int_{-4}^{4} \pi x^2\, dy = 2\pi \int_0^4 (4 - x^2)\, dy = \boxed{2\pi \int_0^4 \left(4 - \frac{y^2}{64}\right) dy} = \frac{128}{5}\pi \text{ cubic units}$$

~~4.~~ 8d Find the volume generated by revolving the area cut off from the parabola $y = 4x - x^2$ by the $x$ axis about the line $y = 6$.

We divide the area vertically (Fig. 41-10). The solid generated by revolving the approximating rectangle about the line $y = 6$ is a washer whose volume is $\pi(6)^2\,\Delta x - \pi(6-y)^2\,\Delta x$. The required volume is then

$$V = \pi \int_0^4 [(6)^2 - (6-y)^2]\,dx = \pi \int_0^4 (12y - y^2)\,dx$$

$$= \boxed{\pi \int_0^4 (48x - 28x^2 + 8x^3 - x^4)\,dx} = \frac{1408\pi}{15} \text{ cubic units}$$

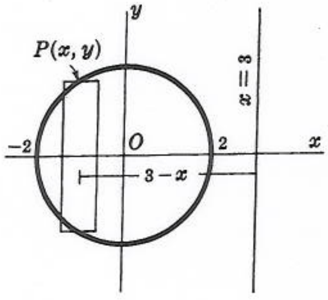


Fig. 41-13

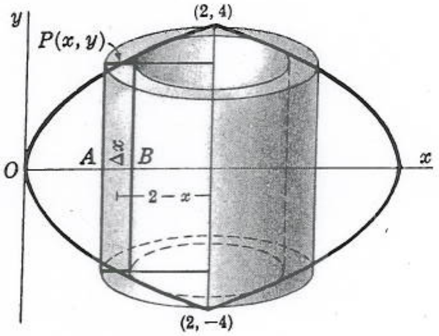


Fig. 41-12

~~6.~~ 8e Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about the latus rectum. Use the shell method. (See Problem 2.)

We divide the area vertically (Fig. 41-12) and, for convenience, choose the point $P$ so that $x$ is the midpoint of the segment $AB$. The approximating rectangle has height $2y = 4\sqrt{2x}$ and width $\Delta x$, and its mean distance from the latus rectum is $2 - x$. When the rectangle is revolved about the latus rectum, the volume of the cylindrical shell generated is $2\pi(2-x)(4\sqrt{2x}\,\Delta x)$. The required volume is then

$$V = \boxed{8\sqrt{2}\pi \int_0^2 (2-x)\sqrt{x}\,dx} = 8\sqrt{2}\pi \int_0^2 (2x^{1/2} - x^{3/2})\,dx = \frac{256\pi}{15} \text{ cubic units}$$

~~7.~~ 8f Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$.

We shall use the shell method (Fig. 41-13). The approximating rectangle is of height $2y$, thickness $\Delta x$, and mean distance from the axis of revolution $3 - x$. The required volume is then

$$V = 2\pi \int_{-2}^{2} 2y(3-x)\,dx = \boxed{4\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2}\,dx} = 12\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx - 4\pi \int_{-2}^{2} x\sqrt{4-x^2}\,dx$$

$$= \left[12\pi\left(\frac{x}{2}\sqrt{4-x^4} + 2\arcsin\frac{x}{2}\right) + \frac{4\pi}{3}(4-x^2)^{3/2}\right]_{-2}^{2} = 24\pi^2 \text{ cubic units}$$

~~9~~.8g Find the volume generated when the plane area bounded by $y = -x^2 - 3x + 6$ and $x + y - 3 = 0$ is revolved (*a*) about $x = 3$, and (*b*) about $y = 0$.

From Fig. 41-15,

(*a*) $V = 2\pi \int_{-3}^{1} (y_C - y_L)(3 - x)\,dx = \boxed{2\pi \int_{-3}^{1} (x^3 - x^2 - 9x + 9)\,dx} = \frac{256\pi}{3}$ cubic units

(*b*) $V = \pi \int_{-3}^{1} y_C^2 - y_L^2\,dx = \boxed{\pi \int_{-3}^{1} (x^4 + 6x^3 - 4x^2 - 30x + 27)\,dx} = \frac{1792\pi}{15}$ cubic units

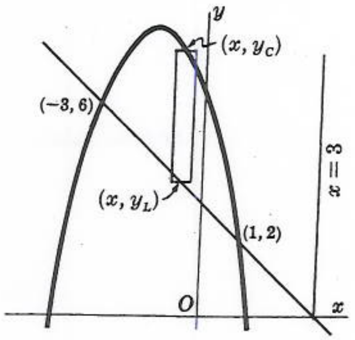


Fig. 41-15

(9a) $W = \int_{x=1}^{x=3} (x^2 + 2x)\,dx = \left[\frac{x^3}{3} + x^2\right]_{x=1}^{x=3} = \frac{50}{3}$ in-lb,

(9b) ↓

(a) Let us introduce a coordinate line $l$ as shown in Figure 6.32, where one end of the spring is attached to some point to the left of the origin and the end to be pulled is located at the origin. According to Hooke's Law (6.14), the force $f(x)$ required to stretch a spring $x$ units beyond its natural length is given by

$$f(x) = kx$$

for some constant $k$. Using the given data, $f(2) = 9$. Substituting in $f(x) = kx$ we obtain $9 = k \cdot 2$, and hence the spring constant is $k = 9/2$. Consequently, for this spring, Hooke's Law has the form

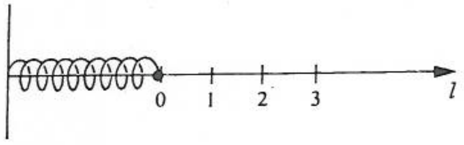
$$f(x) = \frac{9}{2}x.$$

By Definition (6.13) the work done in stretching the spring 4 inches is given by

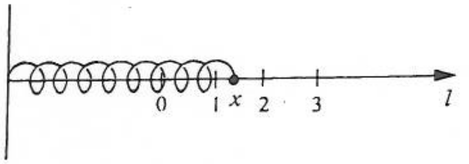
$$W = \int_0^4 \frac{9}{2}x\,dx = \frac{9}{4}x^2\Big]_0^4 = 36 \text{ in.-lb.}$$

(b) We use the same function $f$ but change the interval to $[1, 3]$, obtaining

$$W = \int_1^3 \frac{9}{2}x\,dx = \frac{9}{4}x^2\Big]_1^3 = \frac{81}{4} - \frac{9}{4} = 18 \text{ in.-lb.}$$



Natural length



Stretched $x$ units

*Figure 6.32*

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10. The integral expressing the arc length is boxed.

10a Find the length of the arc of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.

Since $dy/dx = \frac{3}{2}x^{1/2}$,

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \boxed{\int_0^5 \sqrt{1+\frac{9}{4}x}\, dx} = \left[\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\right]_0^5 = \frac{335}{27} \text{ units}$$

10b Find the length of the arc of the curve $x = 3y^{3/2} - 1$ from $y = 0$ to $y = 4$.

Since $dx/dy = \frac{9}{2}y^{1/2}$,

$$s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \boxed{\int_0^4 \sqrt{1+\frac{81}{4}y}\, dy} = \frac{8}{243}(82\sqrt{82}-1) \text{ units}$$

10c Find the length of the arc of $24xy = x^4 + 48$ from $x = 2$ to $x = 4$.

$\frac{dy}{dx} = \frac{x^4-16}{8x^2}$ and $1+\left(\frac{dy}{dx}\right)^2 = \frac{1}{64}\left(\frac{x^4+16}{x^2}\right)^2$. Then $s = \boxed{\frac{1}{8}\int_2^4\left(x^2+\frac{16}{x^2}\right) dx} = \frac{17}{6}$ units.

10d Find the length of the arc of the catenary $y = \frac{1}{2}a(e^{x/a} + e^{-x/a})$ from $x = 0$ to $x = a$.

$\frac{dy}{dx} = \frac{1}{2}(e^{x/a} - e^{-x/a})$ and $1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}(e^{2x/a} - 2 + e^{-2x/a}) = \frac{1}{4}(e^{x/a} + e^{-x/a})^2$. Then

$$\boxed{s = \frac{1}{2}\int_0^a (e^{x/a} + e^{-x/a})\, dx} = \frac{1}{2}a[e^{x/a} - e^{-x/a}]_0^a = \frac{1}{2}a\left(e - \frac{1}{e}\right) \text{ units}$$

10e Find the length of the arc of the curve $x = t^2$, $y = t^3$ from $t = 0$ to $t = 4$.

Here $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$, and $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 9t^4 = 4t^2\left(1+\frac{9}{4}t^2\right)$. Then

$$\boxed{s = \int_0^4 \sqrt{1+\frac{9}{4}t^2}\,(2t\, dt)} = \frac{8}{27}(37\sqrt{37}-1) \text{ units}$$

10f Find the length of an arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

An arch is described as $\theta$ varies from $\theta = 0$ to $\theta = 2\pi$. We have $\frac{dx}{d\theta} = 1 - \cos\theta$, $\frac{dy}{d\theta} = \sin\theta$, and $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 2(1-\cos\theta) = 4\sin^2\frac{1}{2}\theta$. Then $\boxed{s = 2\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta} = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 8$ units.