

MARK BOX	
PROBLEM	POINTS
1 a-j	30
2	10
3	10
4	10
5	10
6	10
7	10
take home	10
%	100

NAME (legibly printed): \_\_\_\_\_

class PIN: 007 \_\_\_\_\_

**INSTRUCTIONS:**

- (1) To receive credit you must:
  - (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that just appears;** such explanations help with partial credit
  - (b) if a line/box is provided, then:
    - show your work BELOW the line/box
    - put your answer on/in the line/box
  - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
- (3) You may **not** use a calculator, books, personal notes.
- (4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
- (5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8<sup>th</sup> ed.):  
 Sections 10.1 - 10.6, 10.8 for the inclass and Section 10.7, 10.9, 10.10 for the take home.

**Problem Inspiration:** See the answer key.

1 ⊕ 2

previous exam; Fall 2006 Exam 3 part 2 # 1 and # 2  
 homework Ch 10 Review # 1-9.

3. previous exams (3a-3c) and HMWK § 10.1 # 21 (3d)

4. Serious Series' Problem # 8.

5. Example from Class

6. Serious Series' Problem # 14

7. Homework problem § 10.10 # 48.

1. Fill-in-the blanks/boxes. All series  $\sum$  are understood to be  $\sum_{n=1}^{\infty}$ .

Hint: I do NOT want to see the words absolute nor conditional on this page!

- 1a.  $n^{\text{th}}$ -term test for an arbitrary series  $\sum a_n$ .

If  $\lim_{n \rightarrow \infty} a_n \neq 0$  or  $\lim_{n \rightarrow \infty} a_n$  does not exist, then  $\sum a_n$  diverges.

- 1b. Geometric Series where  $-\infty < r < \infty$ . The series  $\sum r^n$

- converges if and only if  $|r|$   $< 1$
- diverges if and only if  $|r|$   $\geq 1$

- 1c.  $p$ -series where  $0 < p < \infty$ . The series  $\sum \frac{1}{n^p}$

- converges if and only if  $p$   $> 1$
- diverges if and only if  $p$   $\leq 1$

- 1d. Integral Test for a positive-termed series  $\sum a_n$  where  $a_n \geq 0$ .

Let  $f: [1, \infty) \rightarrow \mathbb{R}$  be so that

- $a_n = f(\underline{n})$  for each  $n \in \mathbb{N}$
- $f$  is a positive function
- $f$  is a continuous function
- $f$  is a nonincreasing (or decreasing) function.

Then  $\sum a_n$  converges if and only if  $\int_1^{\infty} f(x) dx$  converges.

- 1e. Comparison Test for a positive-termed series  $\sum a_n$  where  $a_n \geq 0$ .

- If  $0 \leq a_n \leq b_n$  for all  $n \in \mathbb{N}$  and  $\sum b_n$  converges, then  $\sum a_n$  converges.
- If  $0 \leq b_n \leq a_n$  for all  $n \in \mathbb{N}$  and  $\sum b_n$  diverges, then  $\sum a_n$  diverges.

- 1f. Limit Comparison Test for a positive-termed series  $\sum a_n$  where  $a_n \geq 0$ .

Let  $b_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ .

If  $0 < L < \infty$ , then  $\sum a_n$  converges if and only if  $\sum b_n$  converges.

- 1g. Ratio and Root Tests for a positive-termed series  $\sum a_n$  where  $a_n \geq 0$ .

Let  $\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$  or  $\rho = \lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}}$ .

- If  $\rho$   $< 1$  then  $\sum a_n$  converges.
- If  $\rho$   $> 1$  then  $\sum a_n$  diverges.
- If  $\rho$   $= 1$  then the test is inconclusive.

- 1h. Alternating Series Test for an alternating series  $\sum (-1)^n a_n$  where  $a_n > 0$  for each  $n \in \mathbb{N}$ .

If

- $a_n$   $>$   $a_{n+1}$  for each  $n \in \mathbb{N}$
- $\lim_{n \rightarrow \infty} a_n =$   $0$

then  $\sum (-1)^n a_n$  converges

Ch 10 Review # 7.

1i. By definition, for an arbitrary series  $\sum a_n$ , (fill in the blanks with converges or diverges).

- $\sum a_n$  is absolutely convergent if and only if  $\sum |a_n|$  converges
- $\sum a_n$  is conditionally convergent if and only if  $\sum a_n$  converges and  $\sum |a_n|$  diverges
- $\sum a_n$  is divergent if and only if  $\sum a_n$  diverges

1j. If a power series in  $x - x_0$  has radius of convergence  $R$  where  $0 < R < \infty$ , then the power series is:

- absolutely convergent for  $x \in (x_0 - R, x_0 + R)$  ... i.e. ...  $x_0 - R < x < x_0 + R$  .. i.e.  $|x - x_0| < R$
- divergent for  $x \notin [x_0 - R, x_0 + R]$  ... i.e.  $|x - x_0| > R$

2. Circle T if the statement is TRUE. Circle F if the statement is FALSE. To be more specific: circle T if the statement is always true and circle F if the statement is NOT always true.

Scoring: 2 pts for a correct answer, 1 pt for a blank answer, 0 pts for an incorrect answer.

Ch 10 Review # 9 c/d

- |   |     |     |  |
|---|-----|-----|--|
| { | (T) | F   | If a function $f: [0, \infty) \rightarrow \mathbb{R}$ satisfies that $\lim_{x \rightarrow \infty} f(x) = L$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying that $f(n) = a_n$ for each natural number $n$ , then $\lim_{n \rightarrow \infty} a_n = L$ .   |
|   | T   | (F) | If a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies that $\lim_{n \rightarrow \infty} a_n = L$ and $f: [0, \infty) \rightarrow \mathbb{R}$ is a function satisfying that $f(n) = a_n$ for each natural number $n$ , then $\lim_{x \rightarrow \infty} f(x) = L$ . <i>eg... <math>f(x) = \cos(2\pi x)</math></i> |

Thm 10.4.3  $\rightarrow$  (T)

$a_n = \frac{1}{n}, b_n = -\frac{1}{n} \rightarrow$  T

(T)

- |     |   |
|-----|---|
| F   | If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum (a_n + b_n)$ converges.                     |
| (F) | If $\sum (a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.                     |
| F   | If $S_N = \sum_{n=17}^N r^n$ , then $S_N = \frac{r^{17} - r^{N+1}}{1-r}$ for each $N > 17$ . <i>and</i> |

NOTICE, the above sum starts at  $n = 17$ , not at  $n = 0$ .

$$\begin{aligned}
 S_N &= r^{17} + r^{18} + \dots + r^N \\
 r S_N &= r^{18} + \dots + r^N + r^{N+1} \\
 \hline
 (1-r) S_N &= r^{17} - r^{N+1} \\
 S_N &= \frac{r^{17} - r^{N+1}}{1-r}
 \end{aligned}$$

3. Find the limit of the following sequences. Indicate your reasoning.  
Put ANSWER IN BOX and show WORK BELOW BOX.  
There is: 3a, 3b, 3c (on this page), and then 3d on the next page.

3a.  $\lim_{n \rightarrow \infty} (.999917)^n = 0$

3b.  $\lim_{n \rightarrow \infty} (1.000017)^n = \infty$  or DNE or diverges

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & |r| < 1 \\ \text{DNE} & |r| > 1 \\ 1 & r = 1 \\ \text{DNE} & r = -1 \end{cases}$$

3c.  $\lim_{n \rightarrow \infty} \frac{5n^3 + 6n + 3}{17n^3 + 9n^2 + 4} = \frac{5}{17}$

$$\lim_{n \rightarrow \infty} \frac{5n^3 + 6n + 3}{17n^3 + 9n^2 + 4} = \lim_{n \rightarrow \infty} \frac{5 + \frac{6}{n^2} + \frac{3}{n^3}}{17 + \frac{9}{n} + \frac{4}{n^3}} = \frac{5+0+0}{17+0+0}$$

divide numerator & denom. by  $n^3$  (highest power you see)  $= n^3$ .

3d IS ON NEXT THE PAGE  $\Rightarrow$

3d.  $\lim_{n \rightarrow \infty} \left(\frac{n+6}{n+1}\right)^n = e^5$

Hint: if  $\ln(a_n) \rightarrow 17$ , then  $a_n \equiv e^{\ln(a_n)} \rightarrow e^{17}$

(\*)  $\frac{d}{dn} \left(\frac{n+6}{n+1}\right) = \frac{d}{dn} \left(\frac{n+1+5}{n+1}\right)$   
 $= \frac{d}{dn} (1+5(n+1)^{-1}) = -5(n+1)^{-2}$

$\left(\frac{n+6}{n+1}\right)^n = \left(\frac{1+\frac{6}{n}}{1+\frac{1}{n}}\right)^n \xrightarrow{n \rightarrow \infty} \infty \cdot 0$  indeterminate form.

$\lim_{n \rightarrow \infty} \ln \left(\frac{n+6}{n+1}\right)^n = \lim_{n \rightarrow \infty} n \ln \left(\frac{n+6}{n+1}\right) = \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{n+6}{n+1}\right)}{\frac{1}{n}}$

$\frac{0}{0}$  L'H  $\lim_{n \rightarrow \infty} \frac{D_n \ln \left(\frac{n+6}{n+1}\right)}{D_n \frac{1}{n}} \stackrel{(*)}{=} \lim_{n \rightarrow \infty} \frac{\frac{\frac{n+1}{n+6} \cdot \frac{-5}{(n+1)^2}}{-n^{-2}}}$

$\stackrel{alg}{=} 5 \lim_{n \rightarrow \infty} \frac{n^2}{(n+6)(n+1)} = 5 \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + \textcircled{?}n + \textcircled{??}}$

$= 5 \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{\textcircled{?}}{n} + \frac{\textcircled{??}}{n^2}} = 5 \cdot \frac{1}{1+0+0} = 5$

↑  
see 3c.

Indeterminate Forms - L'Hôpital's Rule

At  $x = u$ ,   has the indeterminate form   if  $\lim_{x \rightarrow u} f(x) =$    and  $\lim_{x \rightarrow u} g(x) =$   

(1)	$\frac{f(x)}{g(x)}$	$\frac{0}{0}$	0	0
(2)	$\frac{f(x)}{g(x)}$	$\frac{\infty}{\infty}$	$\infty$	$\infty$
(3)	$f(x) \cdot g(x)$	$0 \cdot \infty$	0	$\infty$
(4)	$f(x) - g(x)$	$\infty - \infty$	$\infty$	$\infty$
(5)	$[f(x)]^{g(x)}$	$0^0$	0	0
(6)	$[f(x)]^{g(x)}$	$\infty^0$	$\infty$	0
(*) <span style="border: 1px solid black; padding: 2px 5px;">(7)</span>	$[f(x)]^{g(x)}$	$1^\infty$	1	$\infty$

HERE:  $u$  stands for any of the symbols  $a, a^-, a^+, -\infty, +\infty$ .

L'Hôpital's Rule

(1) and (2)

If:

- $\frac{f(x)}{g(x)}$  has the indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  at  $u$

and

- $\lim_{x \rightarrow u} \frac{f'(x)}{g'(x)}$  exists (i.e. this limit is a finite number or  $-\infty$  or  $\infty$ )

then

$$\lim_{x \rightarrow u} \frac{f(x)}{g(x)} = \lim_{x \rightarrow u} \frac{f'(x)}{g'(x)}.$$

(3)

If  $f(x) \cdot g(x)$  has the indeterminate form  $0 \cdot \infty$  at  $u$ , then rewrite:

$$f(x) \cdot g(x) = \frac{f(x)}{1/g(x)}, \text{ which has the indeterminate form } \left[ \frac{0}{0} \right] \text{ at } u$$

or

$$f(x) \cdot g(x) = \frac{g(x)}{1/f(x)}, \text{ which has the indeterminate form } \left[ \frac{\infty}{\infty} \right] \text{ at } u$$

and then apply L'Hôpital's Rule.

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(4)

If  $f(x) - g(x)$  has the indeterminate form  $\boxed{\infty - \infty}$  at  $u$ ,  
then use algebraic manipulation to convert  $f(x) - g(x)$   
into a form of the type  $\boxed{\frac{0}{0}}$  or  $\boxed{\frac{\infty}{\infty}}$   
and then apply L'Hôpital's Rule.

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(5)

If  $[f(x)]^{g(x)}$  has the indeterminate form  $\boxed{0^0}$  at  $u$ , then follow these steps:  
Let

$$y = [f(x)]^{g(x)} .$$

So

$$\ln y = \ln \left( [f(x)]^{g(x)} \right) .$$

Next, simplify

$$\ln y = [g(x)] \cdot \ln [f(x)] .$$

Note that  $\ln y = [g(x)] \cdot \ln [f(x)]$  has the indeterminate form  $\boxed{0 \cdot -\infty}$  at  $u$ .

Using an appropriate above method (i.e. (3)), evaluate

$$\lim_{x \rightarrow u} \ln y \equiv L .$$

Conclude

$$\lim_{x \rightarrow u} \ln [f(x)]^{g(x)} = L \quad \implies \quad \lim_{x \rightarrow u} [f(x)]^{g(x)} = e^L .$$

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(6) and (7)

If  $[f(x)]^{g(x)}$  has the indeterminate form  $\boxed{\infty^0}$  or  $\boxed{1^\infty}$ , then proceed similarly as in (5).

Note that  $\ln y = [g(x)] \cdot \ln [f(x)]$  will have the indeterminate form

(6)  $\boxed{0 \cdot \infty}$  at  $u$

(7)  $\boxed{\infty \cdot 0}$  at  $u$ .

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4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{17n^2 + n - 1}$$

absolutely convergent

conditionally convergent

divergent

$$\lim_{n \rightarrow \infty} (-1)^n \frac{n^2 + 1}{17n^2 + n - 1} \stackrel{\substack{= \\ \uparrow \\ \text{sec 3c}}}{=} \lim_{n \rightarrow \infty} (-1)^n \frac{1 + \frac{1}{n^2}}{17 + \frac{1}{n} - \frac{1}{n^2}} \neq 0$$

b/c it's oscillating.

$n^{\text{th}}$  term test for divg.  $\Rightarrow$  divg.



5. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4 + 1}$$

absolutely convergent

conditionally convergent

divergent

n.b. see class notes.

Ex 2.  $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

- abs. conv.
- cond. conv.
- divg.

Well This is really a beefed-up (made harder) version of Ex 1. Let's learn from (i.e. compare to) Ex 1.

Ex 1.  $\sum (-1)^n \frac{1}{n}$

Ex 2  $\sum (-1)^n \frac{n^3}{n^4+1}$

$\sum (-1)^n \tilde{u}_n$

$\sum (-1)^n u_n$

$u_n = \frac{n^3}{n^4+1} \approx \frac{n^3}{n^4} = \frac{1}{n} = \tilde{u}_n$

So  $\sum (-1)^n u_n$  should behave like  $\sum (-1)^n \tilde{u}_n$ .

$\Downarrow$  Ex 1  
cond. conv.

he should also be cond. conv.  $\leftarrow$

Now. to justify our above guess.

① abs. conv. ?  $\rightarrow$

Consider  $\sum \left| (-1)^n \frac{n^3}{n^4+1} \right| = \sum \frac{n^3}{n^4+1}$

$0 < \frac{n^3}{n^4+1} \approx \frac{n^3}{n^4} = \frac{1}{n} = b_n$

LCT.  $\lim_{n \rightarrow \infty} \frac{\frac{n^3}{n^4+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^4}{n^4+1} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n^4}} = \frac{1}{1+0} = 1$

So  $\sum \frac{n^3}{n^4+1} \not\approx \sum \frac{1}{n}$  "do the same thing."  $0 < 1 < \infty$ .

$\uparrow$  diverges, p-series,  $p=1 \leq 1$

So  $\sum \frac{n^3}{n^4+1}$  diverges

② cond. conv.?  $\rightarrow \square$

Consider  $\sum (-1)^n \frac{n^3}{n^4+1} = \sum (-1)^n u_n$  with  $u_n = \frac{n^3}{n^4+1} > 0$ .

So it's an alternating series

AST. ①  $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n^3}{n^4+1} = \lim_{n \rightarrow \infty} \frac{1/n}{1+1/n^4} = \frac{0}{1+0} = 0 \checkmark$

② Are the  $u_n$ 's decreasing?

$f(x) = \frac{x^3}{x^4+1}$   $\leftarrow$  want dec. so want  $f'(x) < 0$  for  $x \geq 1$  or 17 is good enough  $\downarrow$

$f'(x) \underset{\text{work}}{\overset{\text{Calc I}}{<}} \frac{x^2(3-x^4)}{(x^4+1)^2} < 0 \Leftrightarrow 3-x^4 < 0$

$\Leftrightarrow 3 < x^4$

so dec. for  $n$  big enough.  $\Leftrightarrow \underset{\substack{55 \\ 1.3}}{3^{1/4}} < |x|$

AST.  $\Rightarrow \sum \frac{(-1)^n n^3}{n^4+1}$  conv.

Conclusion

$\sum \left| \frac{(-1)^n n^3}{n^4+1} \right|$  divg

$\sum \frac{(-1)^n n^3}{n^4+1}$  convg.

$\Rightarrow \sum (-1)^n \frac{n^3}{n^4+1}$  is

conditionally convergent.

6. Let

$$a_n = \frac{3^n n!}{(2n)!}$$

6a. Find an expression for  $\frac{a_{n+1}}{a_n}$  that does NOT have a factorial sign (that is a ! sign) in it.

→ Hint:  $(2(n+1))! = (2n+2)!$

$$\frac{a_{n+1}}{a_n} = \frac{3(n+1)}{(2n+1)(2n+2)} \quad \text{or} \quad \frac{3n+3}{4n^2+6n+2}$$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{3^{n+1} (n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{3^n n!} = \frac{3^{n+1}}{3^n} \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+2)!} \\ &= \frac{3^n \cdot 3}{3^n} \frac{n! (n+1)}{n!} \frac{(2n)!}{(2n)! (2n+1)(2n+2)} = \frac{3(n+1)}{(2n+1)(2n+2)} \end{aligned}$$

6b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n n!}{(2n)!}$$



absolutely convergent



conditionally convergent



divergent

abs. conv? consider  $\sum \left| (-1)^n \frac{3^n n!}{(2n)!} \right| = \sum \frac{3^n n!}{(2n)!}$

Factorial  $\rightarrow$  ratio test

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \stackrel{(6a)}{=} \lim_{n \rightarrow \infty} \frac{3n+3}{4n^2+6n+2} \stackrel{\text{see 3c}}{=} \lim_{n \rightarrow \infty} \frac{\frac{3}{n} + \frac{3}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}}$$

$$= \frac{0+0}{4+0+0} = 0 < 1 \Rightarrow \text{conv.}$$

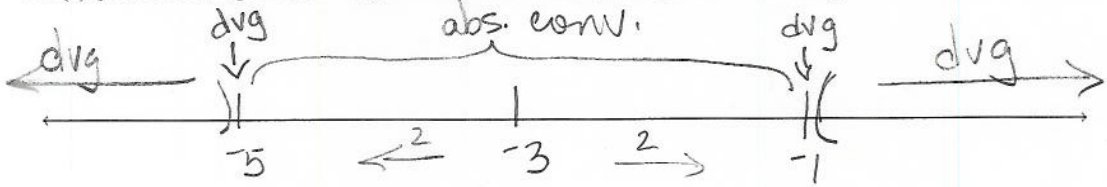
7. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n}$$

Hint:  $(2x+6)^n = [2(x+3)]^n = 2^n(x+3)^n = 2^n(x-(-3))^n$

The center is  $x_0 = \underline{-3}$  and the radius of convergence is  $R = \underline{2}$ .

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



Ratio Test  $\rho = \lim_{n \rightarrow \infty} \left| \frac{(2x+6)^{n+1}}{4^{n+1}} \frac{4^n}{(2x+6)^n} \right| = \lim_{n \rightarrow \infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

↓ or

Root Test  $\rho = \lim_{n \rightarrow \infty} \left| \frac{(2x+6)^n}{4^n} \right|^{1/n} = \lim_{n \rightarrow \infty} \frac{|2x+6|}{4} = \frac{|2x+6|}{4}$

$$\rho < 1 \Leftrightarrow |2x+6| < 4 \Leftrightarrow 2|x+3| < 4 \Leftrightarrow |x+3| < 2 \Leftrightarrow |x-(-3)| < 2$$

endpts.  $-3 + 2 = -1$  and  $-3 - 2 = -5$

Check endpts

$$x = -1 : \sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{4^n}{4^n} = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \dots = \infty \text{ divg}$$

$$x = -5 : \sum_{n=1}^{\infty} \frac{(2x+6)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1 \cdot 4)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

osc btw  $-1$  &  $0 \Rightarrow \text{divg}$