

MARK BOX		
PROBLEM	POINTS	
a - j	10	
TOTAL	10	

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class PIN: 007

**INSTRUCTIONS:**

- (1) To receive credit you must:
  - (a) **work in a logical fashion, show all your work, indicate your reasoning;**  
**no credit will be given for an answer that just appears;**  
such explanations help with partial credit
  - (b) if a line/box is provided, then:
    - show your work BELOW the line/box
    - put your answer on/in the line/box
  - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points.  
Check that your copy of the exam has all of the problems.
- (3) This exam covers (from *Calculus* by Anton, Bivens, Davis 8<sup>th</sup> ed.): § 10.7, 10.9, 10.10 .

**Problem Inspiration:** just like the homework.

**Honor Code Statement**

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Furthermore, I have not only read but will also follow the above Instructions.

I hereby verify that I did NOT receive help from other people on this take-home exam problem.

Signature : \_\_\_\_\_

Taylor/Maclaurin Polynomials and Series

Do parts (a) - (j) for the following problem.

$$f(x) = \frac{1}{1+x} \quad x_0 = 4 \quad J = (2, 6) .$$

You might find it easier to do problems (a) - (j) in a different order. Just do what you find easiest.

- On parts (a) - (i), use ideas from only Sections 10.7 and 10.9, i.e., use only:
  - the definition of Taylor polynomial
  - the definition of Taylor series
  - the theorem/error-estimate on the  $N^{\text{th}}$ -Remainder term for Taylor polynomials.
 Do **NOT** use a known Taylor Series (i.e., do not use methods from Section 10.10).
- On part (j), the very last part, use a known Taylor Series (as from the handout **Commonly Used Taylor Series**) and methods from Section 10.10.

a. Find the following. Note the first column are functions of  $x$  and the second column are numbers.

$f^{(0)}(x) = (1+x)^{-1} = \frac{0!}{1+x}$	$f^{(0)}(x_0) = \frac{1}{5} = (-1)^0 \frac{0!}{5^1}$
$f^{(1)}(x) = - (1+x)^{-2} = - \frac{1!}{(1+x)^2}$	$f^{(1)}(x_0) = - \frac{1}{5^2} = (-1)^1 \frac{1!}{5^2}$
$f^{(2)}(x) = + 2 (1+x)^{-3} = + \frac{2!}{(1+x)^3}$	$f^{(2)}(x_0) = + \frac{2}{5^3} = (-1)^2 \frac{2!}{5^3}$
$f^{(3)}(x) = - 2 \cdot 3 (1+x)^{-4} = - \frac{3!}{(1+x)^4}$	$f^{(3)}(x_0) = - \frac{3!}{5^4} = (-1)^3 \frac{3!}{5^4}$
$f^{(4)}(x) = + 2 \cdot 3 \cdot 4 (1+x)^{-5} = + \frac{4!}{(1+x)^5}$	$f^{(4)}(x_0) = + \frac{4!}{5^5} = (-1)^4 \frac{4!}{5^5}$

b. Find  $N^{\text{th}}$ -order Taylor polynomial of  $y = f(x)$  about  $x_0$  in OPEN form for  $N = 0, 1, 2, 3, 4$ .

$P_0(x) = \frac{1}{5}$
$P_1(x) = \frac{1}{5} - \frac{1}{5^2} (x-4)^1$
$P_2(x) = \frac{1}{5} - \frac{1}{5^2} (x-4)^1 + \frac{1}{5^3} (x-4)^2$
$P_3(x) = \frac{1}{5} - \frac{1}{5^2} (x-4)^1 + \frac{1}{5^3} (x-4)^2 - \frac{1}{5^4} (x-4)^3$
$P_4(x) = \frac{1}{5} - \frac{1}{5^2} (x-4)^1 + \frac{1}{5^3} (x-4)^2 - \frac{1}{5^4} (x-4)^3 + \frac{1}{5^5} (x-4)^4$

- c. Find the Taylor series of  $y = f(x)$  about  $x_0$  in OPEN form.

$$P_{\infty}(x) = \frac{1}{5} - \frac{1}{5^2}(x-4)^1 + \frac{1}{5^3}(x-4)^2 - \frac{1}{5^4}(x-4)^3 + \frac{1}{5^5}(x-4)^4 - \dots$$

- d. Find the Taylor series of  $y = f(x)$  about  $x_0$  in CLOSED form.

$$P_{\infty}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} \cdot (x-4)^n \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{5^{n+1}}$$

- e. Find the  $n^{\text{th}}$  Taylor coefficient of  $y = f(x)$  about  $x_0$ .

$$c_n = \frac{(-1)^n}{5^{n+1}}$$

- f. Find the interval  $I$  of convergence of the Taylor series  $y = f(x)$  about  $x_0$ . Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

$$I = (-1, 9)$$

Ratio Test.

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{(x-4)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(x-4)^n} \right| = \lim_{n \rightarrow \infty} \frac{5^n}{5^{n+1}} \frac{|x-4|^{n+1}}{|x-4|^n}$$

$$= \lim_{n \rightarrow \infty} \frac{|x-4|}{5} = \frac{|x-4|}{5} < 1 \iff |x-4| < 5$$

Check endpoints

$$x=9 \Rightarrow \sum_{n=0}^{\infty} \frac{(x-4)^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{(9-4)^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{5^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{5} = \infty$$

$$x=-1 \Rightarrow \sum_{n=0}^{\infty} \frac{(x-4)^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1-4)^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1 \cdot 5)^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{5^{n+1}}$$

$$\hookrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{5} \text{ divg b/c } \lim_{n \rightarrow \infty} \frac{(-1)^n}{5} \neq 0.$$

$J = (2, 6)$  given at beginning of problem

- g. Consider the given interval  $J$  and fix an  $n \in \mathbb{N}$ . Find a good upper bound for the maximum of  $|f^{(N+1)}(x)|$  on the interval  $J$ . Your answer can have an  $N$  in it but it cannot have an:  $x, x_0, c$ . (Note that  $J$  is a subset of  $I$  but Prof. G. might have picked a smaller  $J$  than  $I$  to make the problem easier.)

$$\max_{c \in J} |f^{(N+1)}(c)| \leq \frac{(N+1)!}{3^{N+2}}$$

$$(a.) \Rightarrow f^{(N)}(x) = \frac{(-1)^N N!}{(1+x)^{N+1}} \Rightarrow$$

$$|f^{(N+1)}(x)| = \left| \frac{(-1)^{N+1} (N+1)!}{(1+x)^{N+2}} \right| = \frac{(N+1)!}{|1+x|^{N+2}} \leq \frac{(N+1)!}{3^{N+2}}$$

$$2 < x < 6$$

$$\Downarrow$$

$$3 < x+1 < 7$$

$$\Downarrow$$

$$3 < |x+1| < 7$$

- h. Consider the given interval  $J$  and fix an  $N \in \mathbb{N}$ . For each  $x \in J$ , find a good upper bound for the maximum of  $|R_N(x)|$ . Your answer can have an  $N$  and  $x$  in it but it cannot have an:  $x_0, c$ .

$$|R_N(x)| \leq \frac{|x-4|^{N+1}}{3^{N+2}} \leq \frac{2^{N+1}}{3^{N+2}}$$

Taylor's Big Theorem  $\Rightarrow$  for some  $c$  between  $x$  and  $4$

$$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} (x-x_0)^{N+1} \right| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} (x-4)^{N+1} \right| \quad \hookrightarrow c \in J$$

$$= |f^{(N+1)}(c)| \cdot \frac{1}{(N+1)!} |x-4|^{N+1}$$

$$\downarrow (g)$$

$$\leq \frac{(N+1)!}{3^{N+2}} \cdot \frac{1}{(N+1)!} |x-4|^{N+1}$$

$$= \frac{|x-4|^{N+1}}{3^{N+2}} \leq \frac{2^{N+1}}{3^{N+2}}$$

$$\uparrow \quad 2 \leq x \leq 6 \Rightarrow -2 \leq x-4 \leq 2$$

- i. Carefully show that  $f(x) = P_\infty(x)$  for each  $x$  in the given interval  $J$  by using part (h) and showing that  $\lim_{N \rightarrow \infty} |R_N(x)| = 0$  for each  $x \in J$ .

$$|R_N(x)| \stackrel{(h)}{\leq} \frac{2^{N+1}}{3^{N+2}}$$

$$= \frac{1}{3} \frac{2^{N+1}}{3^{N+1}}$$

$$= \frac{1}{3} \left(\frac{2}{3}\right)^{N+1} \xrightarrow{N \rightarrow \infty} \frac{1}{3} \cdot 0 = 0$$

b/c

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & |r| < 1 \\ 1 & r = 1 \\ \text{divg} & r = -1 \text{ or } |r| > 1 \end{cases}$$



- j. Using a known Taylor Series (as from the handout Commonly Used Taylor Series) and methods from Section 10.10, find a power series expansion (in CLOSED form) for  $y = f(x)$  about  $x_0$ . Also, say when this power series expansion is valid by examining when the Commonly Used Taylor Series is valid. Show all your work and work in a logical fashion.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} (x-4)^n$$

which is valid for  $|x-4| < 5$   
or  $(-1, 9)$

Going to use Geometric Series  $\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & |r| < 1 \\ \text{divg} & |r| \geq 1 \end{cases}$

$$\frac{1}{1+x} = \frac{1}{5 - [(-1)(x-4)]} = \frac{1}{5} \frac{1}{1 - \left[ \frac{(-1)(x-4)}{5} \right]}$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \left[ \frac{(-1)(x-4)}{5} \right]^n = \sum_{n=0}^{\infty} \frac{1}{5} \frac{(-1)^n (x-4)^n}{5^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} (x-4)^n$$

Here,  $r = \frac{(-1)(x-4)}{5}$  is valid  $\Leftrightarrow |r| < 1$

$$\Leftrightarrow \left| \frac{(-1)(x-4)}{5} \right| < 1 \Leftrightarrow \frac{|x-4|}{5} < 1 \Leftrightarrow |x-4| < 5$$

